

# INTEGRAL AFFINE SCHUR–WEYL RECIPROCITY

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**ABSTRACT.** Let  $\mathfrak{D}_\Delta(n)$  be the double Ringel–Hall algebra of the cyclic quiver  $\Delta(n)$  and let  $\dot{\mathfrak{D}}_\Delta(n)$  be the modified quantum affine algebra of  $\mathfrak{D}_\Delta(n)$ . We will construct an integral form  $\dot{\mathfrak{D}}_\Delta(n)$  for  $\mathfrak{D}_\Delta(n)$  such that the natural algebra homomorphism from  $\dot{\mathfrak{D}}_\Delta(n)$  to the integral affine quantum Schur algebra is surjective. Furthermore, we will use Hall algebras to construct the integral form  $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$  of the universal enveloping algebra  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  of the loop algebra  $\widehat{\mathfrak{gl}}_n = \mathfrak{gl}_n(\mathbb{Q}) \otimes \mathbb{Q}[t, t^{-1}]$ , and prove that the natural algebra homomorphism from  $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$  to the affine Schur algebra over  $\mathbb{Z}$  is surjective.

## 1. INTRODUCTION

The representation of the general linear group  $GL(n, \mathbb{C})$  and the symmetric group  $\mathfrak{S}_r$  over  $\mathbb{C}$  are related by Schur–Weyl reciprocity (cf. [21]). This reciprocity is also true over  $\mathbb{Z}$ . That is, the natural algebra homomorphisms

$$\mathcal{U}_\mathbb{Z}(\mathfrak{gl}_n) \rightarrow \text{End}_{\mathbb{Z}[\mathfrak{S}_r]}(V^{\otimes r}), \quad \mathbb{Z}[\mathfrak{S}_r] \rightarrow \text{End}_{\mathcal{U}_\mathbb{Z}(\mathfrak{gl}_n)}(V^{\otimes r})$$

are surjective, where  $\mathcal{U}_\mathbb{Z}(\mathfrak{gl}_n)$  is the Kostant  $\mathbb{Z}$ -form [15] of the universal enveloping algebra  $\mathcal{U}(\mathfrak{gl}_n)$  of  $\mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{Q})$ , and  $V$  is the natural module for  $\mathcal{U}_\mathbb{Z}(\mathfrak{gl}_n)$  (see [3, 4, 6]). The quantum Schur–Weyl reciprocity between quantum  $\mathfrak{gl}_n$  and Hecke algebras of type  $A$  in the generic case was established in [14] and the integral quantum Schur–Weyl reciprocity was proved in [7, 9]. Furthermore, the cyclotomic Schur–Weyl reciprocity between quantum groups and Ariki–Koike algebras was investigated in [2, 20, 1, 12].

Let  $\mathfrak{D}_\Delta(n)$  be the double Ringel–Hall algebra of the cyclic quiver  $\Delta(n)$  over  $\mathbb{Q}(v)$ , where  $v$  is an indeterminate. Then  $\mathfrak{D}_\Delta(n)$  is isomorphic to the quantum loop algebra  $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$  defined by Drinfeld’s new presentation (cf. [5, 2.3.5]), where  $\widehat{\mathfrak{gl}}_n = \mathfrak{gl}_n \otimes \mathbb{Q}[t, t^{-1}]$  is the loop algebra associated to  $\mathfrak{gl}_n$ . In [5, 3.6.3], it is proved that the natural algebra homomorphism  $\zeta_r$  from  $\mathfrak{D}_\Delta(n)$  to  $\mathcal{S}_\Delta(n, r)$  is surjective, where  $\mathcal{S}_\Delta(n, r)$  is the affine quantum Schur algebra over  $\mathbb{Q}(v)$  (with  $v$  an indeterminate). It is natural to ask whether this result is true over any field. Before discussing this problem, we have to construct a suitable integral form for  $\mathfrak{D}_\Delta(n)$ . Let  $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ . A certain  $\mathcal{Z}$ -submodule of  $\mathfrak{D}_\Delta(n)$ , denoted by  $\mathfrak{D}_\Delta(n)$ , was introduced in [5, (3.8.1.1)], and it is

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conjectured in [5, 3.8.6] that  $\mathfrak{D}_\Delta(n)$  is a  $\mathcal{Z}$ -subalgebra of  $\mathfrak{D}_\Delta(n)$ . If this conjecture is true, then  $\mathfrak{D}_\Delta(n)$  becomes an integral form for  $\mathfrak{D}_\Delta(n)$ .

Let  $\dot{\mathfrak{D}}_\Delta(n)$  be the modified quantum affine algebra of  $\mathfrak{D}_\Delta(n)$ . Associated with  $\mathfrak{D}_\Delta(n)$ , we will construct a certain free  $\mathcal{Z}$ -submodule of  $\dot{\mathfrak{D}}_\Delta(n)$ , denoted by  $\mathfrak{D}_\Delta(n)$ , such that  $\dot{\mathfrak{D}}_\Delta(n) = \mathfrak{D}_\Delta(n) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$ . We will prove in 4.2 and 4.3 that  $\mathfrak{D}_\Delta(n)$  is a  $\mathcal{Z}$ -subalgebra of  $\dot{\mathfrak{D}}_\Delta(n)$  and the natural algebra homomorphism  $\zeta_r$  from  $\mathfrak{D}_\Delta(n)$  to  $\mathcal{S}_\Delta(n, r)$  is surjective, where  $\mathcal{S}_\Delta(n, r)$  is the affine quantum Schur algebra over  $\mathcal{Z}$ .

Let  $\widehat{\mathfrak{D}}_\Delta(n)$  (resp.,  $\widehat{\dot{\mathfrak{D}}}_\Delta(n)$ ) be the completion algebra of  $\mathfrak{D}_\Delta(n)$  (resp.,  $\dot{\mathfrak{D}}_\Delta(n)$ ). We will see in 4.4 and 4.5 that the double Ringel–Hall algebra  $\mathfrak{D}_\Delta(n)$  can be regarded as a subalgebra of  $\widehat{\mathfrak{D}}_\Delta(n)$  and we have a proper inclusion  $\mathfrak{D}_\Delta(n) \subset \widehat{\mathfrak{D}}_\Delta(n) \cap \mathfrak{D}_\Delta(n)$ . Furthermore we will prove in 6.5 that this proper inclusion becomes an equality in the classical case. More precisely, we will use Hall algebras to construct a certain lattice, denoted by  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$ , of the universal enveloping algebra  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  of  $\widehat{\mathfrak{gl}}_n$ . Let  $\widehat{\mathfrak{D}}_\Delta(n)_{\mathbb{Q}}$  (resp.,  $\widehat{\mathfrak{D}}_\Delta(n)_{\mathbb{Z}}$ ) be the completion algebra of  $\dot{\mathfrak{D}}_\Delta(n) \otimes_{\mathcal{Z}} \mathbb{Q}$  (resp.,  $\dot{\mathfrak{D}}_\Delta(n) \otimes_{\mathcal{Z}} \mathbb{Z}$ ), where  $\mathbb{Q}$  and  $\mathbb{Z}$  are regarded as  $\mathcal{Z}$ -modules by specializing  $v$  to 1. We will prove in 6.5 that  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n) = \widehat{\mathfrak{D}}_\Delta(n)_{\mathbb{Z}} \cap \mathcal{U}(\widehat{\mathfrak{gl}}_n)$ . Here  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  is regarded as a subalgebra of  $\widehat{\mathfrak{D}}_\Delta(n)_{\mathbb{Q}}$  via the map  $\varphi$  defined in 6.4. In particular, we conclude that  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  is a  $\mathbb{Z}$ -subalgebra of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  and hence  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  is the integral form of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ . As the quantum affine case, there is a natural surjective algebra homomorphism  $\eta_r : \mathcal{U}(\widehat{\mathfrak{gl}}_n) \rightarrow \mathcal{S}_\Delta(n, r)_{\mathbb{Q}}$ , where  $\mathcal{S}_\Delta(n, r)_{\mathbb{Q}} = \mathcal{S}_\Delta(n, r) \otimes_{\mathcal{Z}} \mathbb{Q}$ . We will prove in 6.7 that the restriction of  $\eta_r$  to  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  yields a surjective  $\mathbb{Z}$ -algebra homomorphism  $\eta_r : \mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n) \rightarrow \mathcal{S}_\Delta(n, r)_{\mathbb{Z}}$ , where  $\mathcal{S}_\Delta(n, r)_{\mathbb{Z}} = \mathcal{S}_\Delta(n, r) \otimes_{\mathcal{Z}} \mathbb{Z}$  (cf. [10]).

We organize this paper as follows. In §2, we will recall the definition of the double Ringel–Hall algebra  $\mathfrak{D}_\Delta(n)$  and the modified quantum affine algebra  $\dot{\mathfrak{D}}_\Delta(n)$  of  $\mathfrak{D}_\Delta(n)$ . We collect in §3 several results concerning affine quantum Schur algebras. In §4 we will construct the  $\mathcal{Z}$ -submodule  $\mathfrak{D}_\Delta(n)$  (resp.,  $\dot{\mathfrak{D}}_\Delta(n)$ ) of  $\mathfrak{D}_\Delta(n)$  (resp.,  $\dot{\mathfrak{D}}_\Delta(n)$ ) and prove in 4.2 that  $\mathfrak{D}_\Delta(n)$  is a  $\mathcal{Z}$ -subalgebra of  $\dot{\mathfrak{D}}_\Delta(n)$ . In addition, we will prove in 4.3 that the natural algebra homomorphism  $\zeta_r$  from  $\mathfrak{D}_\Delta(n)$  to the integral quantum affine Schur algebra  $\mathcal{S}_\Delta(n, r)$  is surjective and establish certain relation between  $\mathfrak{D}_\Delta(n)$  and  $\dot{\mathfrak{D}}_\Delta(n)$  in 4.5. In 5.3, we derive certain commutator formulas in  $\mathfrak{D}_\Delta(n)$ , which will be used in §6. Finally, we will use Hall algebras to introduce the free  $\mathbb{Z}$ -submodule  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ , and prove in 6.5 and 6.7 that  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  is a  $\mathbb{Z}$ -subalgebra of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  such that the natural algebra homomorphism  $\eta_r : \mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n) \rightarrow \mathcal{S}_\Delta(n, r)_{\mathbb{Z}}$  is surjective.

**Notation 1.1.** For a positive integer  $n$ , let  $M_{\Delta, n}(\mathbb{Q})$  be the set of all matrices  $A = (a_{i,j})_{i,j \in \mathbb{Z}}$  with  $a_{i,j} \in \mathbb{Q}$  such that

- (a)  $a_{i,j} = a_{i+n,j+n}$  for  $i, j \in \mathbb{Z}$ ;
- (b) for every  $i \in \mathbb{Z}$ , both sets  $\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}$  and  $\{j \in \mathbb{Z} \mid a_{j,i} \neq 0\}$  are finite.

Let  $\Theta_\Delta(n) = \{A \in M_{\Delta, n}(\mathbb{Q}) \mid a_{i,j} \in \mathbb{N}, \forall i, j\}$ .

Let  $\mathbb{Z}_\Delta^n = \{(\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{Z}, \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z}\}$  and  $\mathbb{N}_\Delta^n = \{(\lambda_i)_{i \in \mathbb{Z}} \in \mathbb{Z}_\Delta^n \mid \lambda_i \geq 0 \text{ for } i \in \mathbb{Z}\}$ . We will identify  $\mathbb{Z}_\Delta^n$  with  $\mathbb{Z}^n$  via the following bijection

$$(1.1.1) \quad \flat : \mathbb{Z}_\Delta^n \longrightarrow \mathbb{Z}^n, \quad \mathbf{j} \longmapsto \flat(\mathbf{j}) = (j_1, \dots, j_n).$$

Let  $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ , where  $v$  is an indeterminate, and let  $\mathbb{Q}(v)$  be the fraction field of  $\mathcal{Z}$ . Specializing  $v$  to 1,  $\mathbb{Q}$  and  $\mathbb{Z}$  will be viewed as  $\mathcal{Z}$ -modules.

## 2. DOUBLE RINGEL–HALL ALGEBRAS OF CYCLIC QUIVERS

Let  $\Delta(n)$  ( $n \geq 2$ ) be the cyclic quiver with vertex set  $I = \mathbb{Z}/n\mathbb{Z} = \{1, 2, \dots, n\}$  and arrow set  $\{i \rightarrow i+1 \mid i \in I\}$ . Let  $\mathbb{F}$  be a field. For  $i \in I$ , let  $S_i$  be the irreducible representation of  $\Delta(n)$  over  $\mathbb{F}$  with  $(S_i)_i = \mathbb{F}$  and  $(S_i)_j = 0$  for  $i \neq j$ . Let

$$\Theta_\Delta^+(n) := \{A \in \Theta_\Delta(n) \mid a_{i,j} = 0 \text{ for } i \geq j\}.$$

For any  $A = (a_{i,j}) \in \Theta_\Delta^+(n)$ , let

$$M(A) = M_\mathbb{F}(A) = \bigoplus_{\substack{1 \leq i \leq n \\ i < j, j \in \mathbb{Z}}} a_{i,j} M^{i,j},$$

where  $M^{i,j}$  is the unique indecomposable representation for  $\Delta(n)$  of length  $j - i$  with top  $S_i$ . For  $A \in \Theta_\Delta^+(n)$  let  $\mathbf{d}(A) \in \mathbb{N}I$  be the dimension vector of  $M(A)$ . We will identify  $\mathbb{N}I$  with  $\mathbb{N}_\Delta^n$  under (1.1.1). By definition we have

$$(2.0.2) \quad \mathbf{d}(A) = \left( \sum_{\substack{s \leq i < t \\ s, t \in \mathbb{Z}}} a_{s,t} \right)_{i \in \mathbb{Z}}$$

for  $A \in \Theta_\Delta^+(n)$ ,

For  $i, j \in \mathbb{Z}$  let  $E_{i,j}^\Delta \in \Theta_\Delta(n)$  be the matrix  $(e_{k,l}^{i,j})_{k,l \in \mathbb{Z}}$  defined by

$$e_{k,l}^{i,j} = \begin{cases} 1 & \text{if } k = i + sn, l = j + sn \text{ for some } s \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

For  $\lambda \in \mathbb{N}_\Delta^n$  let

$$A_\lambda = \sum_{1 \leq i \leq n} \lambda_i E_{i,i+1}^\Delta \in \Theta_\Delta^+(n).$$

Then  $M_\mathbb{F}(A_\lambda)$  is a semisimple representation of  $\Delta(n)$  over  $\mathbb{F}$ .

The Euler form associated with the cyclic quiver  $\Delta(n)$  is the bilinear form  $\langle -, - \rangle : \mathbb{Z}_\Delta^n \times \mathbb{Z}_\Delta^n \rightarrow \mathbb{Z}$  defined by  $\langle \lambda, \mu \rangle = \sum_{1 \leq i \leq n} \lambda_i \mu_i - \sum_{1 \leq i \leq n} \lambda_i \mu_{i+1}$  for  $\lambda, \mu \in \mathbb{Z}_\Delta^n$ .

By [19], for  $A, B, C \in \Theta_\Delta^+(n)$ , there is a polynomial  $\varphi_{A,B}^C \in \mathbb{Z}[v^2]$  such that, for any finite field  $\mathbb{F}_q$ ,  $\varphi_{A,B}^C|_{v^2=q}$  is equal to the number of submodules  $N$  of  $M_{\mathbb{F}_q}(C)$  satisfying  $N \cong M_{\mathbb{F}_q}(B)$  and  $M_{\mathbb{F}_q}(C)/N \cong M_{\mathbb{F}_q}(A)$ .

Let  $\mathfrak{D}_\Delta(n)$  be the double Ringel–Hall algebra of the cyclic quiver of  $\Delta(n)$  (cf. [22] and [5, (2.1.3.2)]). By [5, 2.4.1 and 2.4.4 and 3.9.2] we obtain the following.

**Lemma 2.1.** *The algebra  $\mathfrak{D}_\Delta(n)$  is the algebra over  $\mathbb{Q}(v)$  generated by  $u_A^+$ ,  $K_i^{\pm 1}$ ,  $u_A^-$  ( $A \in \Theta_\Delta^+(n)$ ,  $i \in \mathbb{Z}$ ) subject to the following relations:*

- (1)  $K_i = K_{i+n}$ ,  $K_i K_j = K_j K_i$ ,  $K_i K_i^{-1} = 1$ ,  $u_0^+ = u_0^- = 1$ ;
- (2)  $K^{\mathbf{j}} u_A^+ = v^{\langle \mathbf{d}(A), \mathbf{j} \rangle} u_A^+ K^{\mathbf{j}}$ ,  $u_A^- K^{\mathbf{j}} = v^{\langle \mathbf{d}(A), \mathbf{j} \rangle} K^{\mathbf{j}} u_A^-$ , where  $K^{\mathbf{j}} = K_1^{j_1} \cdots K_n^{j_n}$  for  $\mathbf{j} \in \mathbb{Z}_\Delta^n$ ;
- (3)  $u_A^+ u_B^+ = \sum_{C \in \Theta_\Delta^+(n)} v^{\langle \mathbf{d}(A), \mathbf{d}(B) \rangle} \varphi_{A,B}^C u_C^+$ ;
- (4)  $u_A^- u_B^- = \sum_{C \in \Theta_\Delta^+(n)} v^{\langle \mathbf{d}(B), \mathbf{d}(A) \rangle} \varphi_{B,A}^C u_C^-$ ;
- (5) commutator relations: for all  $\lambda, \mu \in \mathbb{N}_\Delta^n$ ,

$$v^{\langle \mu, \mu \rangle} \sum_{\substack{\alpha, \beta \in \mathbb{N}_\Delta^n \\ \lambda - \alpha = \mu - \beta \geq 0}} \varphi_{\lambda, \mu}^{\alpha, \beta} v^{\langle \beta, \lambda + \mu - \beta \rangle} \tilde{K}^{\mu - \beta} u_{A_\beta}^- u_{A_\alpha}^+ = v^{\langle \mu, \lambda \rangle} \sum_{\substack{\alpha, \beta \in \mathbb{N}_\Delta^n \\ \lambda - \alpha = \mu - \beta \geq 0}} \varphi_{\lambda, \mu}^{\alpha, \beta} v^{\langle \mu - \beta, \alpha \rangle + \langle \mu, \beta \rangle} \tilde{K}^{\beta - \mu} u_{A_\alpha}^+ u_{A_\beta}^-,$$

where  $\tilde{K}^\nu := (\tilde{K}_1)^{\nu_1} \cdots (\tilde{K}_n)^{\nu_n}$  with  $\tilde{K}_i = K_i K_{i+1}^{-1}$  for  $\nu \in \mathbb{Z}_\Delta^n$ , and

$$\varphi_{\lambda, \mu}^{\alpha, \beta} = v^{2 \sum_{1 \leq i \leq n} (\lambda_i - \alpha_i)(1 - \alpha_i - \beta_i)} \prod_{\substack{1 \leq i \leq n \\ 0 \leq s \leq \lambda_i - \alpha_i - 1}} \frac{1}{v^{2(\lambda_i - \alpha_i)} - v^{2s}}.$$

Let  $\mathbf{U}_\Delta(n)$  be the subalgebra of  $\mathfrak{D}_\Delta(n)$  generated by  $u_{E_{i,i+1}^\Delta}^+$ ,  $u_{E_{i+1,i}^\Delta}^-$  and  $K_i^{\pm 1}$  for  $1 \leq i \leq n$ . The algebra  $\mathbf{U}_\Delta(n)$  is a proper subalgebra of  $\mathfrak{D}_\Delta(n)$  and it is the quantum affine algebra considered in [17, 7.7].

Let  $\mathfrak{D}_\Delta^+(n) = \text{span}_{\mathbb{Q}(v)}\{u_A^+ \mid A \in \Theta_\Delta^+(n)\}$ ,  $\mathfrak{D}_\Delta^-(n) = \text{span}_{\mathbb{Q}(v)}\{u_A^- \mid A \in \Theta_\Delta^+(n)\}$ , and  $\mathfrak{D}_\Delta^0(n) = \text{span}_{\mathbb{Q}(v)}\{K^{\mathbf{j}} \mid \mathbf{j} \in \mathbb{Z}_\Delta^n\}$ . Then we have

$$(2.1.1) \quad \mathfrak{D}_\Delta(n) \cong \mathfrak{D}_\Delta^+(n) \otimes \mathfrak{D}_\Delta^0(n) \otimes \mathfrak{D}_\Delta^-(n).$$

For  $i \in \mathbb{Z}$  let  $e_i^\Delta \in \mathbb{N}_\Delta^n$  be the element satisfying  $\mathbf{b}(e_i^\Delta) = \mathbf{e}_i = (0, \dots, 0, \frac{1}{(i)}, 0, \dots, 0)$ , where  $\mathbf{b}$  is defined in (1.1.1). Let  $\Pi_\Delta(n) = \{\alpha_j^\Delta := e_j^\Delta - e_{j+1}^\Delta \mid 1 \leq j \leq n\}$ . By 2.1 the algebra  $\mathfrak{D}_\Delta(n)$  is a  $\mathbb{Z}_\Delta^n$ -graded algebra with

$$\deg(u_A^+) = \sum_{1 \leq i \leq n} d_i \alpha_i^\Delta, \quad \deg(u_A^-) = - \sum_{1 \leq i \leq n} d_i \alpha_i^\Delta \quad \text{and} \quad \deg(K_i^{\pm 1}) = 0$$

for  $A \in \Theta_\Delta^+(n)$  and  $1 \leq i \leq n$ , where  $(d_i)_{i \in \mathbb{Z}} = \mathbf{d}(A)$ . For  $\nu \in \mathbb{Z}_\Delta^n$  let  $\mathfrak{D}_\Delta(n)_\nu$  be the set of homogeneous elements in  $\mathfrak{D}_\Delta(n)$  of degree  $\nu$ . Then we have

$$\mathfrak{D}_\Delta(n) = \bigoplus_{\nu \in \mathbb{Z} \Pi_\Delta(n)} \mathfrak{D}_\Delta(n)_\nu.$$

**Lemma 2.2.** *For  $\lambda, \mathbf{j} \in \mathbb{Z}_\Delta^n$  and  $t \in \mathfrak{D}_\Delta(n)_\lambda$  we have  $K^{\mathbf{j}} t = v^{\mathbf{j} \cdot \lambda} t K^{\mathbf{j}}$ , where  $\lambda \cdot \mathbf{j} = \sum_{1 \leq j \leq n} \lambda_j j_i$ .*

*Proof.* Clearly for  $A \in \Theta_\Delta^+(n)$  we have  $\langle \mathbf{d}(A), \mathbf{j} \rangle = (\deg u_A^+) \cdot \mathbf{j}$  and  $-\langle \mathbf{d}(A), \mathbf{j} \rangle = (\deg u_A^-) \cdot \mathbf{j}$ . Combining this with 2.1(2) proves the assertion.  $\square$

Following [16] we now introduce the modified quantum affine algebra  $\dot{\mathfrak{D}}_\Delta(n)$  of  $\mathfrak{D}_\Delta(n)$ . For  $\lambda, \mu \in \mathbb{Z}_\Delta^n$  we set

$${}_\lambda \mathfrak{D}_\Delta(n)_\mu = \mathfrak{D}_\Delta(n) \Big/ \left( \sum_{\mathbf{j} \in \mathbb{Z}_\Delta^n} (K^{\mathbf{j}} - v^{\lambda \cdot \mathbf{j}}) \mathfrak{D}_\Delta(n) + \sum_{\mathbf{j} \in \mathbb{Z}_\Delta^n} \mathfrak{D}_\Delta(n) (K^{\mathbf{j}} - v^{\mu \cdot \mathbf{j}}) \right).$$

Let  $\pi_{\lambda, \mu} : \mathfrak{D}_\Delta(n) \rightarrow {}_\lambda \mathfrak{D}_\Delta(n)_\mu$  be the canonical projection. Let

$$\dot{\mathfrak{D}}_\Delta(n) := \bigoplus_{\lambda, \mu \in \mathbb{Z}_\Delta^n} {}_\lambda \mathfrak{D}_\Delta(n)_\mu.$$

Since  ${}_\lambda \mathfrak{D}_\Delta(n)_\mu = \bigoplus_{\nu \in \mathbb{Z}\Pi_\Delta(n)} \pi_{\lambda, \mu}(\mathfrak{D}_\Delta(n)_\nu)$ , we have  $\dot{\mathfrak{D}}_\Delta(n) = \bigoplus_{\nu \in \mathbb{Z}\Pi_\Delta(n)} \dot{\mathfrak{D}}_\Delta(n)_\nu$ , where  $\dot{\mathfrak{D}}_\Delta(n)_\nu = \bigoplus_{\lambda, \mu \in \mathbb{Z}_\Delta^n} \pi_{\lambda, \mu}(\mathfrak{D}_\Delta(n)_\nu)$ .

**Lemma 2.3.** *Assume  $\lambda, \mu \in \mathbb{Z}_\Delta^n$ ,  $\nu \in \mathbb{Z}\Pi_\Delta(n)$  and  $\nu \neq \lambda - \mu$ . Then we have  $\pi_{\lambda, \mu}(\mathfrak{D}_\Delta(n)_\nu) = 0$ .*

*Proof.* Let  $t \in \mathfrak{D}_\Delta(n)_\nu$ . By 2.2 we see that  $(v^{\lambda_i} - v^{\mu_i + \nu_i})\pi_{\lambda, \mu}(t) = \pi_{\lambda, \mu}(K_i t) - \pi_{\lambda, \mu}(v^{\nu_i} t K_i) = 0$  for  $1 \leq i \leq n$ . Since  $\nu \neq \lambda - \mu$  and  $v$  is an indeterminate, there exist  $1 \leq i_0 \leq n$  such that  $v^{\lambda_{i_0}} \neq v^{\mu_{i_0} + \nu_{i_0}}$ . Consequently,  $\pi_{\lambda, \mu}(t) = 0$ .  $\square$

We define the product in  $\dot{\mathfrak{D}}_\Delta(n)$  as follows. For  $\lambda', \mu', \lambda'', \mu'' \in \mathbb{Z}_\Delta^n$  with  $\lambda' - \mu', \lambda'' - \mu'' \in \mathbb{Z}\Pi_\Delta(n)$  and any  $t \in \mathfrak{D}_\Delta(n)_{\lambda' - \mu'}, s \in \mathfrak{D}_\Delta(n)_{\lambda'' - \mu''}$ , define

$$\pi_{\lambda', \mu'}(t) \pi_{\lambda'', \mu''}(s) = \begin{cases} \pi_{\lambda', \mu''}(ts), & \text{if } \mu' = \lambda'' \\ 0 & \text{otherwise.} \end{cases}$$

Then by 2.3 one can check that  $\dot{\mathfrak{D}}_\Delta(n)$  becomes an associative  $\mathbb{Q}(v)$ -algebra structure with the above product.

The algebra  $\dot{\mathfrak{D}}_\Delta(n)$  is naturally a  $\mathfrak{D}_\Delta(n)$ -bimodule defined by

$$(2.3.1) \quad t' \pi_{\lambda', \lambda''}(s) t'' = \pi_{\lambda' + \nu', \lambda'' - \nu''}(t' s t'')$$

for  $t' \in \mathfrak{D}_\Delta(n)_{\nu'}$ ,  $s \in \mathfrak{D}_\Delta(n)$ ,  $t'' \in \mathfrak{D}_\Delta(n)_{\nu''}$  and  $\lambda', \lambda'' \in \mathbb{Z}_\Delta^n$ .

### 3. AFFINE QUANTUM SCHUR ALGEBRAS

For  $r \geq 0$  let  $\mathcal{S}_\Delta(n, r)^1$  be the affine quantum Schur algebra over  $\mathbb{Q}(v)$  defined in [17, 1.9]. Recall the set  $\Theta_\Delta(n)$  defined in 1.1. The algebra  $\mathcal{S}_\Delta(n, r)$  has a normalized  $\mathbb{Q}(v)$ -basis  $\{[A] \mid A \in \Theta_\Delta(n, r)\}$  (cf. [17, 1.9]), where

$$\Theta_\Delta(n, r) = \{A \in \Theta_\Delta(n) \mid \sigma(A) := \sum_{\substack{1 \leq i \leq n \\ j \in \mathbb{Z}}} a_{i, j} = r\}.$$

Let  $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ , where  $v$  is an indeterminate. Let  $\mathcal{S}_\Delta(n, r)$  be the  $\mathcal{Z}$ -submodule of  $\mathcal{S}_\Delta(n, r)$  spanned by  $\{[A] \mid A \in \Theta_\Delta(n, r)\}$ . Then  $\mathcal{S}_\Delta(n, r)$  is the  $\mathcal{Z}$ -subalgebra of  $\mathcal{S}_\Delta(n, r)$ .

<sup>1</sup>The algebra  $\mathcal{S}_\Delta(n, r)$  is denoted by  $\mathfrak{U}_{r, n, n}$  in [17, 1.9].

For  $r \geq 0$ , let  $\Lambda_\Delta(n, r) = \{\lambda \in \mathbb{N}_\Delta^n \mid \sigma(\lambda) := \sum_{1 \leq i \leq n} \lambda_i = r\}$ . For  $\lambda \in \Lambda_\Delta(n, r)$  and  $A \in \Theta_\Delta(n, r)$ , we have

$$(3.0.2) \quad [\text{diag}(\lambda)] \cdot [A] = \begin{cases} [A], & \text{if } \lambda = \text{ro}(A); \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad [A][\text{diag}(\lambda)] = \begin{cases} [A], & \text{if } \lambda = \text{co}(A); \\ 0, & \text{otherwise,} \end{cases}$$

where  $\text{ro}(A) = (\sum_{j \in \mathbb{Z}} a_{i,j})_{i \in \mathbb{Z}}$  and  $\text{co}(A) = (\sum_{i \in \mathbb{Z}} a_{i,j})_{j \in \mathbb{Z}}$  (see [17, 1.9]). In particular, we have

$$(3.0.3) \quad [\text{diag}(\lambda)][\text{diag}(\mu)] = \delta_{\lambda,\mu}[\text{diag}(\lambda)]$$

for  $\lambda, \mu \in \Lambda_\Delta(n, r)$ .

We now recall certain triangular relation in  $\mathcal{S}_\Delta(n, r)$ , which will be needed in §3. First we need the following order relation  $\preccurlyeq$  on  $\Theta_\Delta(n)$ . For  $A \in \Theta_\Delta(n)$  and  $i \neq j \in \mathbb{Z}$ , let

$$\sigma_{i,j}(A) = \sum_{s \leq i, t \geq j} a_{s,t} \text{ if } i < j, \text{ and } \sigma_{i,j}(A) = \sum_{s \geq i, t \leq j} a_{s,t} \text{ if } i > j.$$

For  $A, B \in \Theta_\Delta(n)$ , define  $B \preccurlyeq A$  if  $\sigma_{i,j}(B) \leq \sigma_{i,j}(A)$  for all  $i \neq j$ . Put  $B \prec A$  if  $B \preccurlyeq A$  and, for some pair  $(i, j)$  with  $i \neq j$ ,  $\sigma_{i,j}(B) < \sigma_{i,j}(A)$ .

Let  $\Theta_\Delta^\pm(n) := \{A \in \Theta_\Delta(n) \mid a_{i,j} = 0 \text{ for } i = j\}$ . For  $A \in \Theta_\Delta^\pm(n)$  and  $\mathbf{j} \in \mathbb{Z}_\Delta^n$ , define  $A(\mathbf{j}, r) \in \mathcal{S}_\Delta(n, r)$  by

$$A(\mathbf{j}, r) = \begin{cases} \sum_{\lambda \in \Lambda_\Delta(n, r - \sigma(A))} v^{\lambda \cdot \mathbf{j}} [A + \text{diag}(\lambda)], & \text{if } \sigma(A) \leq r; \\ 0, & \text{otherwise.} \end{cases}$$

For  $A \in \Theta_\Delta^\pm(n)$ , write  $A = A^+ + A^-$  with  $A^+ \in \Theta_\Delta^+(n)$ ,  $A^- \in \Theta_\Delta^-(n)$ , where

$$\Theta_\Delta^-(n) := \{A \in \Theta_\Delta(n) \mid a_{i,j} = 0 \text{ for } i \leq j\}.$$

The following triangular relation in  $\mathcal{S}_\Delta(n, r)$  is given in [5, 3.7.3].

**Lemma 3.1.** *Let  $C \in \Theta_\Delta^\pm(n)$ . Then the following triangular relation holds in  $\mathcal{S}_\Delta(n, r)$ :*

$$C^+(\mathbf{0}, r)C^-(\mathbf{0}, r) = C(\mathbf{0}, r) + \sum_{\substack{X \in \Theta_\Delta^\pm(n) \\ X \prec C, \mathbf{j} \in \mathbb{Z}_\Delta^n}} h_{C,X,\mathbf{j};r} X(\mathbf{j}, r),$$

where  $h_{C,X,\mathbf{j};r} \in \mathbb{Q}(v)$ .

Using 3.1 one can construct a  $\mathcal{Z}$ -basis for  $\mathcal{S}_\Delta(n, r)$  as follows.

**Corollary 3.2** ([5, 3.7.7]). *The set  $\{A^+(\mathbf{0}, r)[\text{diag}(\lambda)]A^-(\mathbf{0}, r) \mid A \in \Theta_\Delta^\pm(n), \lambda \in \Lambda_\Delta(n, r), \lambda_i \geq \sigma_i(A), \text{ for } 1 \leq i \leq n\}$  forms a  $\mathcal{Z}$ -basis for  $\mathcal{S}_\Delta(n, r)$ , where  $\sigma_i(A) = \sum_{j < i} (a_{i,j} + a_{j,i})$ .*

The double Ringel–Hall algebra  $\mathfrak{D}_\Delta(n)$  is related to the affine quantum Schur algebra in the following way (cf. [11, 17]).

**Lemma 3.3.** [5, 3.6.3] *For  $r \geq 0$ , there is a surjective algebra homomorphism  $\zeta_r : \mathfrak{D}_\Delta(n) \rightarrow \mathcal{S}_\Delta(n, r)$  such that*

$$\zeta_r(K^{\mathbf{j}}) = 0(\mathbf{j}, r), \quad \zeta_r(\tilde{u}_A^+) = A(\mathbf{0}, r), \quad \text{and} \quad \zeta_r(\tilde{u}_A^-) = ({}^t A)(\mathbf{0}, r),$$

for all  $\mathbf{j} \in \mathbb{Z}_\Delta^n$  and  $A \in \Theta_\Delta^+(n)$ , where  ${}^t A$  is the transpose matrix of  $A$  and  $\tilde{u}_A^\pm = {}_{\nu^{\dim \text{End}(M(A)) - \dim M(A)}} u_A^\pm$ .

For convenience, we set  $[A] = 0 \in \mathcal{S}_\Delta(n, r)$  for  $A \notin \Theta_\Delta(n, r)$ . Then we have the following commutation formula in  $\mathcal{S}_\Delta(n, r)$ .

**Lemma 3.4.** *Let  $\lambda \in \mathbb{Z}_\Delta^n$  and  $\nu \in \mathbb{Z}\Pi_\Delta(n)$ . If  $t \in \mathfrak{D}_\Delta(n)_\nu$ , then*

$$\zeta_r(t)[\text{diag}(\lambda)] = [\text{diag}(\lambda + \nu)]\zeta_r(t).$$

*Proof.* Applying (3.0.2) gives

$$(3.4.1) \quad C(\mathbf{0}, r)[\text{diag}(\lambda)] = [C + \text{diag}(\lambda - \text{co}(C))] = [\text{diag}(\lambda + \text{ro}(C) - \text{co}(C))]C(\mathbf{0}, r)$$

for  $C \in \Theta_\Delta^\pm(n)$  and  $\lambda \in \mathbb{Z}_\Delta^n$ . Furthermore by (2.0.2), we conclude that

$$(3.4.2) \quad \deg(u_A^+) = \text{ro}(A) - \text{co}(A) \quad \text{and} \quad \deg(u_A^-) = \text{co}(A) - \text{ro}(A)$$

for  $A \in \Theta_\Delta^+(n)$ . Combining (3.4.1) and (3.4.2) shows that  $\zeta_r(\tilde{u}_A^+)[\text{diag}(\lambda)] = [\text{diag}(\lambda + \deg(\tilde{u}_A^+))]\zeta_r(u_A^+)$  and  $\zeta_r(\tilde{u}_A^-)[\text{diag}(\lambda)] = [\text{diag}(\lambda + \deg(\tilde{u}_A^-))]\zeta_r(\tilde{u}_A^-)$ . This finishes the proof.  $\square$

Finally, we prove that the map  $\zeta_r : \mathfrak{D}_\Delta(n) \rightarrow \mathcal{S}_\Delta(n, r)$  induces a natural algebra homomorphism  $\dot{\zeta}_r : \dot{\mathfrak{D}}_\Delta(n) \rightarrow \mathcal{S}_\Delta(n, r)$ .

**Lemma 3.5.** *There is an algebra homomorphism  $\dot{\zeta}_r : \dot{\mathfrak{D}}_\Delta(n) \rightarrow \mathcal{S}_\Delta(n, r)$  such that*

$$\dot{\zeta}_r(\pi_{\lambda, \mu}(u)) = [\text{diag}(\lambda)]\zeta_r(u)[\text{diag}(\mu)]$$

for  $u \in \mathfrak{D}_\Delta(n)$  and  $\lambda, \mu \in \mathbb{Z}_\Delta^n$ .

*Proof.* Clearly we have

$$[\text{diag}(\lambda)]\zeta_r\left(\sum_{\mathbf{j} \in \mathbb{Z}_\Delta^n} (K^{\mathbf{j}} - v^{\lambda \cdot \mathbf{j}})\mathfrak{D}_\Delta(n) + \sum_{\mathbf{j} \in \mathbb{Z}_\Delta^n} \mathfrak{D}_\Delta(n)(K^{\mathbf{j}} - v^{\mu \cdot \mathbf{j}})\right)[\text{diag}(\mu)] = 0$$

for  $\lambda, \mu \in \mathbb{Z}_\Delta^n$ . Thus  $\dot{\zeta}_r$  is well defined.

Assume  $\lambda', \mu', \lambda'', \mu'' \in \mathbb{Z}_\Delta^n$  is such that  $\lambda' - \mu', \lambda'' - \mu'' \in \mathbb{Z}\Pi_\Delta(n)$ . Let  $t \in \mathfrak{D}_\Delta(n)_{\lambda' - \mu'}$  and  $s \in \mathfrak{D}_\Delta(n)_{\lambda'' - \mu''}$ . If  $\mu' \neq \lambda''$ , then by (3.0.3),  $\dot{\zeta}_r(\pi_{\lambda', \mu'}(t)\pi_{\lambda'', \mu''}(s)) = 0 = \dot{\zeta}_r(\pi_{\lambda', \mu'}(t))\dot{\zeta}_r(\pi_{\lambda'', \mu''}(s))$ . If  $\mu' = \lambda''$ , then by (3.0.3) and 3.4,  $\dot{\zeta}_r(\pi_{\lambda', \mu'}(t))\dot{\zeta}_r(\pi_{\mu', \mu''}(s)) = [\text{diag}(\lambda')]\zeta_r(t)\zeta_r(s)[\text{diag}(\mu'')] = \dot{\zeta}_r(\pi_{\lambda', \mu''}(ts)) = \dot{\zeta}_r(\pi_{\lambda', \mu'}(t)\pi_{\mu', \mu''}(s))$ .  $\square$

Recall that  $\dot{\mathfrak{D}}_\Delta(n)$  is a  $\mathfrak{D}_\Delta(n)$ -bimodule defined by (2.3.1).

**Lemma 3.6.** *We have  $\dot{\zeta}_r(u_1 u_2 u_3) = \zeta_r(u_1) \dot{\zeta}_r(u_2) \zeta_r(u_3)$  for  $u_1, u_3 \in \mathfrak{D}_\Delta(n)$  and  $u_2 \in \dot{\mathfrak{D}}_\Delta(n)$ .*

*Proof.* By 3.4, for  $\lambda, \mu, \nu', \nu'' \in \mathbb{Z}_\Delta^n$  and  $u' \in \mathfrak{D}_\Delta(n)_{\nu'}$ ,  $u'' \in \mathfrak{D}_\Delta(n)_{\nu''}$ ,  $u \in \mathfrak{D}_\Delta(n)_{\lambda-\mu}$ , we have  $\zeta_r(u') \dot{\zeta}_r(\pi_{\lambda,\mu}(u)) \zeta_r(u'') = \zeta_r(u') \zeta_r(u) [\text{diag}(\mu)] \zeta_r(u'') = \zeta_r(u') \zeta_r(u) \zeta_r(u'') [\text{diag}(\mu - \nu'')] = \dot{\zeta}_r(\pi_{\lambda+\nu', \mu-\nu''}(u' u u'')) = \dot{\zeta}_r(u' \pi_{\lambda\mu}(u) u'')$ .  $\square$

#### 4. THE INTEGRAL FORM $\dot{\mathfrak{D}}_\Delta(n)$ OF $\mathfrak{D}_\Delta(n)$

Let  $\mathfrak{D}_\Delta^+(n) = \text{span}_{\mathcal{Z}}\{\tilde{u}_A^+ \mid A \in \Theta_\Delta^+(n)\}$ ,  $\mathfrak{D}_\Delta^-(n) = \text{span}_{\mathcal{Z}}\{\tilde{u}_A^- \mid A \in \Theta_\Delta^-(n)\}$ , and let  $\mathfrak{D}_\Delta^0(n)$  be the  $\mathcal{Z}$ -subalgebra of  $\mathfrak{D}_\Delta(n)$  generated by  $K_i^{\pm 1}$  and  $\begin{bmatrix} K_i; 0 \\ t \end{bmatrix}$  for  $1 \leq i \leq n$  and  $t > 0$ , where  $\begin{bmatrix} K_i; 0 \\ t \end{bmatrix} = \prod_{s=1}^t \frac{K_i v^{-s+1} - K_i^{-1} v^{s-1}}{v^s - v^{-s}}$ . Let  $\mathfrak{D}_\Delta(n) = \mathfrak{D}_\Delta^+(n) \mathfrak{D}_\Delta^0(n) \mathfrak{D}_\Delta^-(n)$  and let

$$(4.0.1) \quad \dot{\mathfrak{D}}_\Delta(n) = \text{span}_{\mathcal{Z}}\{\tilde{u}_A^+ 1_\lambda \tilde{u}_B^- \mid A, B \in \Theta_\Delta^+(n), \lambda \in \mathbb{Z}_\Delta^n\} \subseteq \dot{\mathfrak{D}}_\Delta(n),$$

where  $1_\lambda = \pi_{\lambda,\lambda}(1)$ . Clearly by definition we have

$$\dot{\mathfrak{D}}_\Delta(n) = \bigoplus_{\lambda, \mu \in \mathbb{Z}_\Delta^n} \pi_{\lambda,\mu}(\mathfrak{D}_\Delta(n)).$$

Furthermore by (2.1.1), the set  $\{\tilde{u}_A^+ 1_\lambda \tilde{u}_B^- \mid A, B \in \Theta_\Delta^+(n), \lambda \in \mathbb{Z}_\Delta^n\}$  forms a  $\mathcal{Z}$ -basis for  $\dot{\mathfrak{D}}_\Delta(n)$ .

In [5, 3.8.6], it is conjectured that  $\mathfrak{D}_\Delta(n)$  is a  $\mathcal{Z}$ -subalgebra of  $\mathfrak{D}_\Delta(n)$ . We will prove in 4.2 that the modified version  $\dot{\mathfrak{D}}_\Delta(n)$  of  $\mathfrak{D}_\Delta(n)$  is a  $\mathcal{Z}$ -subalgebra of  $\dot{\mathfrak{D}}_\Delta(n)$ , and prove in 4.3 that the restriction of  $\dot{\zeta}_r : \dot{\mathfrak{D}}_\Delta(n) \rightarrow \mathcal{S}_\Delta(n, r)$  to  $\dot{\mathfrak{D}}_\Delta(n)$  gives a surjective algebra homomorphism from  $\dot{\mathfrak{D}}_\Delta(n)$  to  $\mathcal{S}_\Delta(n, r)$ . Furthermore we will establish certain relation between  $\mathfrak{D}_\Delta(n)$  and  $\dot{\mathfrak{D}}_\Delta(n)$  in (4.5.1).

For  $A, B \in \Theta_\Delta^+(n)$  we write

$$(4.0.2) \quad \tilde{u}_B^- \tilde{u}_A^+ = \sum_{\substack{C \in \Theta_\Delta^\pm(n) \\ \mathbf{j} \in \mathbb{Z}_\Delta^n, j_n=0}} g_{A,B,C,\mathbf{j}} \tilde{u}_{C^+}^+ \tilde{u}_{\mathfrak{t}(C^-)}^- \tilde{K}_1^{j_1} \cdots \tilde{K}_{n-1}^{j_{n-1}}$$

where  $\mathfrak{t}(C^-)$  is the transpose matrix of  $C^-$  and  $g_{A,B,C,\mathbf{j}} \in \mathbb{Q}(v)$

For  $A, B \in \Theta_\Delta^+(n)$ ,  $C \in \Theta_\Delta^\pm(n)$  and  $\lambda \in \mathbb{Z}_\Delta^n$ , let

$$f_{A,B,C,\lambda} = \sum_{\mathbf{j} \in \mathbb{Z}_\Delta^n, j_n=0} g_{A,B,C,\mathbf{j}} v^{(\lambda_1 - \lambda_2)j_1 + \cdots + (\lambda_{n-1} - \lambda_n)j_{n-1}}.$$

**Lemma 4.1.** *Fix  $A, B \in \Theta_\Delta^+(n)$  and  $\lambda \in \mathbb{Z}_\Delta^n$ . Then we have  $f_{A,B,C,\lambda} \in \mathcal{Z}$  for  $C \in \Theta_\Delta^\pm(n)$ .*

*Proof.* Let  $\mathcal{I} = \{C \in \Theta_\Delta^\pm(n) \mid f_{A,B,C,\lambda} \neq 0\}$ . Then  $\mathcal{I}$  is a finite set. It is enough to prove  $\mathcal{J} := \{C \in \mathcal{I} \mid f_{A,B,C,\lambda} \notin \mathcal{Z}\} = \emptyset$ . Suppose this is not the case. We choose a maximal element  $C_0$  in  $\mathcal{J}$  with respect to  $\preccurlyeq$ . Then  $f_{A,B,C_0,\lambda} \notin \mathcal{Z}$ . Furthermore, we choose  $m > 0$  such that  $\lambda + m\mathbf{1} \geq \text{co}(C)$  for all  $C \in \mathcal{I}$ . Let  $\mu = \lambda + m\mathbf{1} \in \mathbb{N}_\Delta^n$ , where  $\mathbf{1} = (\cdots, 1, \cdots, 1, \cdots) \in \Lambda_\Delta(n, n)$ . Let  $r = \sigma(\mu) \geq 0$ .



Applying (3.0.3) gives

$$\begin{aligned}\zeta_r(\tilde{K}_1^{j_1} \cdots \tilde{K}_{n-1}^{j_{n-1}})[\text{diag}(\mu)] &= v^{(\mu_1 - \mu_2)j_1 + \cdots + (\mu_{n-1} - \mu_n)j_{n-1}}[\text{diag}(\mu)] \\ &= v^{(\lambda_1 - \lambda_2)j_1 + \cdots + (\lambda_{n-1} - \lambda_n)j_{n-1}}[\text{diag}(\mu)].\end{aligned}$$

Combining this with 3.3 and (4.0.2) yields

$$\begin{aligned}\zeta_r(\tilde{u}_B^- \tilde{u}_A^+)[\text{diag}(\mu)] &= \sum_{\substack{C \in \Theta_\Delta^\pm(n) \\ \mathbf{j} \in \mathbb{Z}_\Delta^n, j_n = 0}} g_{A,B,C,\mathbf{j}} C^+(\mathbf{0}, r) C^-(\mathbf{0}, r) \zeta_r(\tilde{K}_1^{j_1} \cdots \tilde{K}_{n-1}^{j_{n-1}})[\text{diag}(\mu)] \\ &= \sum_{C \in \Theta_\Delta^\pm(n)} f_{A,B,C,\lambda} C^+(\mathbf{0}, r) C^-(\mathbf{0}, r) [\text{diag}(\mu)].\end{aligned}$$

Since  $\zeta_r(\tilde{u}_B^- \tilde{u}_A^+)[\text{diag}(\mu)] = ({}^t B)(\mathbf{0}, r) A(\mathbf{0}, r) [\text{diag}(\mu)] \in \mathcal{S}_\Delta(n, r)$  we conclude that

$$\begin{aligned}(4.1.1) \quad Y &:= \zeta_r(\tilde{u}_B^- \tilde{u}_A^+)[\text{diag}(\mu)] - \sum_{\substack{C \in \Theta_\Delta^\pm(n) \\ C \notin \mathcal{J}}} f_{A,B,C,\lambda} C^+(\mathbf{0}, r) C^-(\mathbf{0}, r) [\text{diag}(\mu)] \\ &= \sum_{C \in \mathcal{J}} f_{A,B,C,\lambda} C^+(\mathbf{0}, r) C^-(\mathbf{0}, r) [\text{diag}(\mu)] \in \mathcal{S}_\Delta(n, r).\end{aligned}$$

It follows from 3.1 that

$$\begin{aligned}Y &= \sum_{C \in \mathcal{J}} f_{A,B,C,\lambda} \left( C(\mathbf{0}, r) + \sum_{\substack{X \in \Theta_\Delta^\pm(n) \\ X \prec C, \mathbf{j} \in \mathbb{Z}_\Delta^n}} h_{C,X,\mathbf{j};r} X(\mathbf{j}, r) \right) [\text{diag}(\mu)] \\ &= \sum_{C \in \mathcal{J}} f_{A,B,C,\lambda} \left( \sum_{\substack{X \in \Theta_\Delta^\pm(n) \\ X \preccurlyeq C}} t_{C,X} [X + \text{diag}(\mu - \text{co}(X))] \right) \\ &= \sum_{X \in \Theta_\Delta^\pm(n)} l_X [X + \text{diag}(\mu - \text{co}(X))]\end{aligned}$$

where  $t_{C,C} = 1$ ,  $t_{C,X} = \sum_{\mathbf{j} \in \mathbb{Z}_\Delta^n} h_{C,X,\mathbf{j};r} v^{\mathbf{j} \cdot (\mu - \text{co}(X))}$  for  $X \prec C$  and

$$l_X = \sum_{\substack{C \in \mathcal{J} \\ X \preccurlyeq C}} f_{A,B,C,\lambda} t_{C,X}.$$

This, together with (4.1.1) and the fact that the set  $\{[T] \mid T \in \Theta_\Delta(n, r)\}$  forms a  $\mathcal{Z}$ -basis for  $\mathcal{S}_\Delta(n, r)$ , implies that  $l_X \in \mathcal{Z}$  for all  $X \in \Theta_\Delta^\pm(n)$ . Thus  $f_{A,B,C_0,\lambda} = l_{C_0} \in \mathcal{Z}$ . This is a contradiction.  $\square$

**Theorem 4.2.** *The  $\mathcal{Z}$ -module  $\dot{\mathfrak{D}}_\Delta(n)$  is a  $\mathcal{Z}$ -subalgebra of  $\dot{\mathfrak{D}}_\Delta(n)$ . Thus  $\dot{\mathfrak{D}}_\Delta(n)$  is the integral form for  $\dot{\mathfrak{D}}_\Delta(n)$ .*

*Proof.* By (4.0.2) and 4.1 we conclude that

$$\begin{aligned} (\tilde{u}_{A_2}^+ 1_\lambda \tilde{u}_{B_2}^-)(\tilde{u}_{A_1}^+ 1_\mu \tilde{u}_{B_1}^-) &= \sum_{\substack{C \in \Theta_\Delta^\pm(n) \\ j \in \mathbb{Z}_\Delta^n, jn=0}} g_{A_1, B_2, C, j}(\tilde{u}_{A_2}^+ 1_\lambda \tilde{u}_{C^+}^+)(\tilde{u}_{C^-}^- \tilde{K}_1^{j_1} \cdots \tilde{K}_{n-1}^{j_{n-1}} 1_\mu \tilde{u}_{B_1}^-) \\ &= \sum_{C \in \Theta_\Delta^\pm(n)} f_{A_1, B_2, C, \mu}(\tilde{u}_{A_2}^+ 1_\lambda \tilde{u}_{C^+}^+)(\tilde{u}_{C^-}^- 1_\mu \tilde{u}_{B_1}^-) \in \dot{\mathfrak{D}}_\Delta(n). \end{aligned}$$

for  $A_1, A_2, B_1, B_2 \in \Theta_\Delta^+(n)$  and  $\lambda, \mu \in \mathbb{Z}_\Delta^n$ . This proves the assertion.  $\square$

**Corollary 4.3.** *By restriction, the map  $\dot{\zeta}_r$  defined in 3.5 induces a surjective algebra homomorphism  $\dot{\zeta}_r : \dot{\mathfrak{D}}_\Delta(n) \rightarrow \mathcal{S}_\Delta(n, r)$ .*

*Proof.* Applying 3.6 yields

$$\dot{\zeta}_r(\dot{\mathfrak{D}}_\Delta(n)) = \text{span}_{\mathcal{Z}}\{A^+(\mathbf{0}, r)[\text{diag}(\lambda)]A^-(\mathbf{0}, r) \mid A \in \Theta_\Delta^\pm(n), \lambda \in \Lambda_\Delta(n, r)\}.$$

Combining this with 3.2 and 4.2 proves the assertion.  $\square$

We end this section by studying the relation between  $\mathfrak{D}_\Delta(n)$  and  $\dot{\mathfrak{D}}_\Delta(n)$ . We define the completion algebra  $\hat{\mathfrak{D}}_\Delta(n)$  of  $\dot{\mathfrak{D}}_\Delta(n)$  as follows. Let  $\hat{\mathfrak{D}}_\Delta(n)$  be the vector space of all formal (possibly infinite)  $\mathbb{Q}(v)$ -linear combinations  $\sum_{A, B \in \Theta_\Delta^+(n), \lambda \in \mathbb{Z}_\Delta^n} \beta_{A, B, \lambda} u_A^+ 1_\lambda u_B^-$  satisfying

(F): for any  $\mu \in \mathbb{Z}^n$ , the sets  $\{(A, B, \lambda) \mid A, B \in \Theta_\Delta^+(n), \lambda \in \mathbb{Z}_\Delta^n, \beta_{A, B, \lambda} \neq 0, \lambda - \deg(u_B^-) = \mu\}$  and  $\{(A, B, \lambda) \mid A, B \in \Theta_\Delta^+(n), \lambda \in \mathbb{Z}_\Delta^n, \beta_{A, B, \lambda} \neq 0, \lambda + \deg(u_A^+) = \mu\}$  are finite.

We define the product on  $\hat{\mathfrak{D}}_\Delta(n)$  by

$$\sum_{A, B, \lambda} \beta_{A, B, \lambda} u_A^+ 1_\lambda u_B^- \sum_{A', B', \lambda'} \gamma_{A', B', \lambda'} u_{A'}^+ 1_{\lambda'} u_{B'}^- = \sum_{\substack{A, B, \lambda \\ A', B', \lambda'}} \beta_{A, B, \lambda} \gamma_{A', B', \lambda'} (u_A^+ 1_\lambda u_B^-)(u_{A'}^+ 1_{\lambda'} u_{B'}^-)$$

where  $(u_A^+ 1_\lambda u_B^-)(u_{A'}^+ 1_{\lambda'} u_{B'}^-)$  is the product in  $\dot{\mathfrak{D}}_\Delta(n)$ . Since  $(u_A^+ 1_\lambda u_B^-)(u_{A'}^+ 1_{\lambda'} u_{B'}^-)$  is a linear combination of elements  $u_X^+ 1_\mu u_Y^-$  such that  $\lambda + \deg(u_A^+) = \mu + \deg(u_X^+)$  and  $\lambda' - \deg(u_{B'}^-) = \mu - \deg(u_Y^-)$ , the right hand side of the above equation is a well defined elements in  $\dot{\mathfrak{D}}_\Delta(n)$ . In this way,  $\hat{\mathfrak{D}}_\Delta(n)$  becomes an associative algebra. The element  $\sum_{\lambda \in \mathbb{Z}_\Delta^n} 1_\lambda$  is the unit element of  $\hat{\mathfrak{D}}_\Delta(n)$ .

Similarly, we may define the completion algebra  $\hat{\mathfrak{D}}_\Delta(n)$  of  $\dot{\mathfrak{D}}_\Delta(n)$ . Then by definition,  $\hat{\mathfrak{D}}_\Delta(n)$  is the  $\mathcal{Z}$ -submodule of  $\hat{\mathfrak{D}}_\Delta(n)$  consisting of those elements  $\sum_{A, B \in \Theta_\Delta^+(n), \lambda \in \mathbb{Z}_\Delta^n} \beta_{A, B, \lambda} u_A^+ 1_\lambda u_B^-$  in  $\hat{\mathfrak{D}}_\Delta(n)$  with  $\beta_{A, B, \lambda} \in \mathcal{Z}$  for all  $A, B, \lambda$ .

**Proposition 4.4.** *The map  $\Phi : \mathfrak{D}_\Delta(n) \rightarrow \hat{\mathfrak{D}}_\Delta(n)$  defined by sending  $u$  to  $\sum_{\lambda \in \mathbb{Z}_\Delta^n} u 1_\lambda$  for  $u \in \mathfrak{D}_\Delta(n)$  is an injective algebra homomorphism.*

*Proof.* Let  $u \in \mathfrak{D}_\Delta(n)_\lambda$ ,  $w \in \mathfrak{D}_\Delta(n)_\mu$ , where  $\lambda, \mu \in \mathbb{Z}\Pi_\Delta(n)$ . Since  $1_\alpha w = w1_{\alpha-\mu}$  for  $\alpha \in \mathbb{Z}_\Delta^n$ , we have

$$\Phi(u)\Phi(w) = \sum_{\alpha, \beta \in \mathbb{Z}_\Delta^n} (u1_\alpha)(w1_\beta) = \sum_{\alpha, \beta \in \mathbb{Z}_\Delta^n} uw(1_{\alpha-\mu}1_\beta) = \sum_{\beta \in \mathbb{Z}_\Delta^n} uw1_\beta = \Phi(uw).$$

Thus  $\Phi$  is an algebra homomorphism.

Now let us prove that  $\Phi$  is injective. Assume  $x = \sum_{A, B \in \Theta_\Delta^+(n), \mathbf{j} \in \mathbb{Z}_\Delta^n} \beta_{A, B, \mathbf{j}} u_A^+ K^{\mathbf{j}} u_B^- \in \ker(\Phi)$ , where  $\beta_{A, B, \mathbf{j}} \in \mathbb{Q}(v)$ . Then we have

$$\begin{aligned} \Phi(x) &= \sum_{\substack{A, B \in \Theta_\Delta^+(n) \\ \mathbf{j}, \lambda \in \mathbb{Z}_\Delta^n}} \beta_{A, B, \mathbf{j}} u_A^+ K^{\mathbf{j}} 1_{\lambda + \deg(u_B^-)} u_B^- \\ &= \sum_{\substack{A, B \in \Theta_\Delta^+(n) \\ \lambda \in \mathbb{Z}_\Delta^n}} \left( \sum_{\mathbf{j} \in \mathbb{Z}_\Delta^n} \beta_{A, B, \mathbf{j}} v^{(\lambda + \deg(u_B^-)) \cdot \mathbf{j}} \right) u_A^+ 1_{\lambda + \deg(u_B^-)} u_B^-. \end{aligned}$$

This implies that  $\sum_{\mathbf{j} \in \mathbb{Z}_\Delta^n} \beta_{A, B, \mathbf{j}} v^{(\lambda + \deg(u_B^-)) \cdot \mathbf{j}} = 0$  for  $A, B \in \Theta_\Delta^+(n)$  and  $\lambda \in \mathbb{Z}_\Delta^n$ . By the proof of [8, 5.2] we see that  $\det(v^{\mu \cdot \mathbf{j}})_{\mu, \mathbf{j} \in \mathbb{Z}_{[a, b]}^n} \neq 0$  for  $a < b$ , where  $\mathbb{Z}_{[a, b]}^n = \{\mathbf{x} \in \mathbb{Z}^n \mid a \leq x_i \leq b, \text{ for } 1 \leq i \leq n\}$ . It follows that  $\beta_{A, B, \mathbf{j}} = 0$  for all  $A, B, \mathbf{j}$  and hence  $x = 0$ . The proof is completed.  $\square$

**Remark 4.5.** With the above proposition we may regard  $\mathfrak{D}_\Delta(n)$  as a subalgebra of  $\widehat{\mathfrak{D}}_\Delta(n)$ . Clearly, we have

$$(4.5.1) \quad \mathfrak{D}_\Delta(n) \subseteq \widehat{\mathfrak{D}}_\Delta(n) \cap \mathfrak{D}_\Delta(n).$$

Note that  $\mathfrak{D}_\Delta(n) \neq \widehat{\mathfrak{D}}_\Delta(n) \cap \mathfrak{D}_\Delta(n)$ . For example we have  $\frac{K_1 K_2}{v-1} - \frac{K_1}{v-1} \notin \mathfrak{D}_\Delta(n)$ . But  $\frac{K_1 K_2}{v-1} - \frac{K_1}{v-1} = \sum_{\lambda \in \mathbb{Z}_\Delta^n} v^{\lambda_1} \frac{v^{\lambda_2} - 1}{v-1} 1_\lambda \in \widehat{\mathfrak{D}}_\Delta(n) \cap \mathfrak{D}_\Delta(n)$ . Although  $\mathfrak{D}_\Delta(n) \neq \widehat{\mathfrak{D}}_\Delta(n) \cap \mathfrak{D}_\Delta(n)$ , we will prove in 6.5 that the proper inclusion (4.5.1) becomes an equality in the classical case.

## 5. THE COMMUTATOR FORMULAS FOR $\mathfrak{D}_\Delta(n)$

By [5, 1.4.3],  $\mathfrak{D}_\Delta(n)$  is generated by  $u_{E_{i,j}^\Delta}^+$ ,  $u_{E_{i,j}^\Delta}^-$  and  $K_i$ , for  $i, j \in \mathbb{Z}$  and  $i < j$ . Note that  $M(E_{i,j}^\Delta)$  is the indecomposable representation of  $\Delta(n)$  for all  $i < j$ . We derive commutator formulas between indecomposable generators of  $\mathfrak{D}_\Delta(n)$ , which will be used in §6.

For  $A = (a_{i,j}) \in \Theta_\Delta^+(n)$ , there is a polynomial  $\mathfrak{a}_A = \mathfrak{a}_A(v^2) \in \mathcal{Z}$  in  $v^2$  such that, for each finite field  $\mathbb{F}$  with  $q$  elements,  $\mathfrak{a}_A|_{v^2=q} = |\text{Aut}(M_{\mathbb{F}}(A))|$  (see [18, Cor. 2.1.1]). Furthermore, for  $A = (a_{i,j}) \in \Theta_\Delta^+(n)$ , set

$$\mathfrak{d}(A) = \sum_{i < j, 1 \leq i \leq n} a_{i,j}(j-i).$$

Then  $\dim_{\mathbb{F}} M(A) = \mathfrak{d}(A)$  for each finite field  $\mathbb{F}$ . For  $A, B \in \Theta_{\Delta}^+(n)$ , let

$$(5.0.2) \quad \begin{aligned} L_{A,B} &= v^{\langle \mathfrak{d}(B), \mathfrak{d}(B) \rangle} \sum_{A_1, B_1} \varphi_{A,B}^{A_1, B_1} v^{\langle \mathfrak{d}(B_1), \mathfrak{d}(A) + \mathfrak{d}(B) - \mathfrak{d}(B_1) \rangle} \widetilde{K}^{\mathfrak{d}(B) - \mathfrak{d}(B_1)} u_{B_1}^- u_{A_1}^+, \\ R_{A,B} &= v^{\langle \mathfrak{d}(B), \mathfrak{d}(A) \rangle} \sum_{A_1, B_1} \widetilde{\varphi_{A,B}^{A_1, B_1}} v^{\langle \mathfrak{d}(B) - \mathfrak{d}(B_1), \mathfrak{d}(A_1) \rangle + \langle \mathfrak{d}(B), \mathfrak{d}(B_1) \rangle} \widetilde{K}^{\mathfrak{d}(B_1) - \mathfrak{d}(B)} u_{A_1}^+ u_{B_1}^- \end{aligned}$$

where

$$\begin{aligned} \varphi_{A,B}^{A_1, B_1} &= \frac{\mathfrak{a}_{A_1} \mathfrak{a}_{B_1}}{\mathfrak{a}_A \mathfrak{a}_B} \sum_{A_2 \in \Theta_{\Delta}^+(n)} v^{2\mathfrak{d}(A_2)} \mathfrak{a}_{A_2} \varphi_{A_1, A_2}^A \varphi_{B_1, A_2}^B, \\ \widetilde{\varphi_{A,B}^{A_1, B_1}} &= \frac{\mathfrak{a}_{A_1} \mathfrak{a}_{B_1}}{\mathfrak{a}_A \mathfrak{a}_B} \sum_{A_2 \in \Theta_{\Delta}^+(n)} v^{2\mathfrak{d}(A_2)} \mathfrak{a}_{A_2} \varphi_{A_2, A_1}^A \varphi_{A_2, B_1}^B. \end{aligned}$$

By [5, 2.4.4] we have the following result.

**Lemma 5.1.** *For all  $A, B \in \Theta_{\Delta}^+(n)$ ,  $L_{A,B} = R_{A,B}$ .*

For  $s < t$  we let

$$m_{s,t} = \left| \left\{ c \in \mathbb{Z} \mid 0 \leq c \leq \frac{t-s-1}{n} \right\} \right| - 1 = \left\lfloor \frac{t-s-1}{n} \right\rfloor,$$

and set  $m_{s,s} = 0$ . For  $i \in \mathbb{Z}$ , let  $\bar{i}$  denote the integer modulo  $n$ .

**Lemma 5.2.** *Let  $A = E_{i,j}^{\Delta}$ ,  $B = E_{k,l}^{\Delta}$  with  $i < j$  and  $k < l$ . Assume  $A_1, B_1 \in \Theta_{\Delta}^+(n)$ .*

(1) *If  $\varphi_{A,B}^{A_1, B_1} \neq 0$ , then one of the following holds.*

- (i) *If  $A_1 = A$  and  $B_1 = B$ , then  $\varphi_{A,B}^{A_1, B_1} = 1$ .*
- (ii) *If either  $A_1 \neq A$  or  $B_1 \neq B$ , then  $\bar{j} = \bar{l}$  and*

$$\varphi_{A,B}^{A_1, B_1} = \begin{cases} (v^2 - 1)v^{2a_s} & \text{if } A_1 = E_{i,s}^{\Delta} \text{ and } B_1 = E_{k,s-j+l}^{\Delta} \text{ for } \max\{i, k-l+j\} < s < j, \\ (v^2 - 1)^{-1}v^{2a_i} & \text{if } k-l+j = i \text{ and } A_1 = B_1 = 0, \\ v^{2a_i} & \text{if } k-l+j < i, A_1 = 0 \text{ and } B_1 = E_{k,i-j+l}^{\Delta}, \\ v^{2a_{k-l+j}} & \text{if } k-l+j > i, A_1 = E_{i,k-l+j}^{\Delta} \text{ and } B_1 = 0, \end{cases}$$

where  $a_s = m_{i,s} + m_{k-l+j,s} - m_{i,j} - m_{k,l} + m_{s,j} + j - s$ , for  $\max\{i, k-l+j\} \leq s \leq j$ .

(2) *If  $\widetilde{\varphi_{A,B}^{A_1, B_1}} \neq 0$ , then one of the following holds.*

- (i) *If  $A_1 = A$  and  $B_1 = B$ , then  $\widetilde{\varphi_{A,B}^{A_1, B_1}} = 1$ .*

(ii) If either  $A_1 \neq A$  or  $B_1 \neq B$ , then  $\bar{i} = \bar{k}$  and

$$\widetilde{\varphi_{A,B}^{A_1,B_1}} = \begin{cases} (v^2 - 1)v^{2b_s} & \text{if } A_1 = E_{s,j}^\Delta \text{ and } B_1 = E_{s,l-k+i}^\Delta \text{ for } i < s < \min\{j, l-k+i\}, \\ (v^2 - 1)^{-1}v^{2b_j} & \text{if } l-k+i = j \text{ and } A_1 = B_1 = 0, \\ v^{2b_j} & \text{if } l-k+i > j, A_1 = 0 \text{ and } B_1 = E_{j,l-k+i}^\Delta, \\ v^{2b_{l-k+i}} & \text{if } l-k+i < j, A_1 = E_{l-k+i,j}^\Delta \text{ and } B_1 = 0, \end{cases}$$

where  $b_s = m_{s,j} + m_{s+k-i,l} - m_{i,j} - m_{k,l} + m_{i,s} + s - i$ , for  $i \leq s \leq \min\{j, l-k+i\}$ .

*Proof.* Applying [5, (1.2.0.9)] yields  $\mathbf{a}_{E_{s,t}^\Delta} = v^{2m_{s,t}}(v^2 - 1)$  for  $s < t$ . Furthermore if  $A_1, A_2 \neq 0 \in \Theta_\Delta^+(n)$ , then

$$\varphi_{A_1,A_2}^{E_{s,t}^\Delta} = \begin{cases} 1 & A_1 = E_{s,x}^\Delta \text{ and } A_2 = E_{x,t}^\Delta \text{ for some } s < x < t, \\ 0 & \text{otherwise.} \end{cases}$$

since the module  $M(E_{s,t}^\Delta)$  is uniserial. Now the assertion follows from the definition of  $\varphi_{A,B}^{A_1,B_1}$ .  $\square$

For convenience, we let  $\mathbf{d}(E_{i,i}^\Delta) = 0$  for any  $i \in \mathbb{Z}$ . Fix  $i < j$  and  $k < l$ . For  $\max\{i, k-l+j\} \leq s \leq j$  let

$$\begin{aligned} f_s &= 2a_s + \langle \mathbf{d}(E_{k,l}^\Delta), \mathbf{d}(E_{k,l}^\Delta) \rangle + \langle \mathbf{d}(E_{k,s-j+l}^\Delta), \mathbf{d}(E_{i,j}^\Delta) + \mathbf{d}(E_{s,j}^\Delta) \rangle \\ \widetilde{f}_s &= f_s - f_j = 2a_s + \langle \mathbf{d}(E_{k,s-j+l}^\Delta), \mathbf{d}(E_{s,j}^\Delta) \rangle - \langle \mathbf{d}(E_{s-j+l,l}^\Delta), \mathbf{d}(E_{i,j}^\Delta) \rangle \end{aligned}$$

where  $a_s$  is as in 5.2. Furthermore for  $i \leq s \leq \min\{j, l-k+i\}$  let

$$\begin{aligned} g_s &= 2b_s + \langle \mathbf{d}(E_{k,l}^\Delta), \mathbf{d}(E_{i,j}^\Delta) \rangle + \langle \mathbf{d}(E_{i,s}^\Delta), \mathbf{d}(E_{s,j}^\Delta) \rangle + \langle \mathbf{d}(E_{k,l}^\Delta), \mathbf{d}(E_{s+k-i,l}^\Delta) \rangle \\ \widetilde{g}_s &= g_s - g_i = 2b_s + \langle \mathbf{d}(E_{i,s}^\Delta), \mathbf{d}(E_{s,j}^\Delta) \rangle - \langle \mathbf{d}(E_{k,l}^\Delta), \mathbf{d}(E_{k,s+k-i}^\Delta) \rangle \end{aligned}$$

where  $b_s$  is as in 5.2. We can now prove the following commutator formulas in  $\mathfrak{D}_\Delta(n)$ .

**Proposition 5.3.** Assume  $i, j \in \mathbb{Z}$ ,  $i < j$  and  $k < l$ .

- (1) If  $\bar{j} \neq \bar{l}$  and  $\bar{i} \neq \bar{k}$ , then  $u_{E_{i,j}^\Delta}^+ u_{E_{k,l}^\Delta}^- = u_{E_{k,l}^\Delta}^- u_{E_{i,j}^\Delta}^+$ .
- (2) Assume  $\bar{j} = \bar{l}$  and  $\bar{i} \neq \bar{k}$ .

(i) If  $k-l < i-j$ , then

$$u_{E_{i,j}^\Delta}^+ u_{E_{k,l}^\Delta}^- - u_{E_{k,l}^\Delta}^- u_{E_{i,j}^\Delta}^+ = (v^2 - 1) \sum_{i < s < j} v^{\widetilde{f}_s} K_s K_j^{-1} u_{E_{k,s-j+l}^\Delta}^- u_{E_{i,s}^\Delta}^+ + v^{\widetilde{f}_i} K_i K_j^{-1} u_{E_{k,i-j+l}^\Delta}^-.$$

(ii) If  $k-l > i-j$ , then

$$u_{E_{i,j}^\Delta}^+ u_{E_{k,l}^\Delta}^- - u_{E_{k,l}^\Delta}^- u_{E_{i,j}^\Delta}^+ = (v^2 - 1) \sum_{k-l+j < s < j} v^{\widetilde{f}_s} K_s K_j^{-1} u_{E_{k,s-j+l}^\Delta}^- u_{E_{i,s}^\Delta}^+ + v^{\widetilde{f}_{k-l+j}} K_k K_j^{-1} u_{E_{i,k-l+j}^\Delta}^+.$$

(3) Assume  $\bar{j} \neq \bar{l}$  and  $\bar{i} = \bar{k}$ .

(i) If  $k - l < i - j$ , then

$$u_{E_{i,j}^\Delta}^+ u_{E_{k,l}^\Delta}^- - u_{E_{k,l}^\Delta}^- u_{E_{i,j}^\Delta}^+ = (1 - v^2) \sum_{i < s < j} v^{\tilde{g}_s} K_s K_i^{-1} u_{E_{s,j}^\Delta}^+ u_{E_{s+k-i,l}^\Delta}^- - v^{\tilde{g}_j} K_j K_i^{-1} u_{E_{j+k-i,l}^\Delta}^-.$$

(ii) If  $k - l > i - j$ , then

$$u_{E_{i,j}^\Delta}^+ u_{E_{k,l}^\Delta}^- - u_{E_{k,l}^\Delta}^- u_{E_{i,j}^\Delta}^+ = (1 - v^2) \sum_{i < s < l-k+i} v^{\tilde{g}_s} K_s K_i^{-1} u_{E_{s,j}^\Delta}^+ u_{E_{s+k-i,l}^\Delta}^- - v^{\tilde{g}_{l-k+i}} K_l K_i^{-1} u_{E_{l-k+i,j}^\Delta}^+.$$

(4) Assume  $\bar{j} = \bar{l}$  and  $\bar{i} = \bar{k}$ .

(i) If  $k - l = i - j$ , then  $E_{i,j}^\Delta = E_{k,l}^\Delta$  and

$$\begin{aligned} u_{E_{i,j}^\Delta}^+ u_{E_{i,j}^\Delta}^- - u_{E_{i,j}^\Delta}^- u_{E_{i,j}^\Delta}^+ &= (v^2 - 1) \sum_{i < s < j} (v^{\tilde{f}_s} K_s K_j^{-1} u_{E_{i,s}^\Delta}^- u_{E_{i,s}^\Delta}^+ - v^{\tilde{g}_s} K_s K_i^{-1} u_{E_{s,j}^\Delta}^+ u_{E_{s,j}^\Delta}^-) \\ &\quad + \frac{K_i K_j^{-1} - K_i^{-1} K_j}{v^2 - 1} v^{\tilde{f}_i}. \end{aligned}$$

(ii) If  $k - l < i - j$ , then

$$\begin{aligned} u_{E_{i,j}^\Delta}^+ u_{E_{k,l}^\Delta}^- - u_{E_{k,l}^\Delta}^- u_{E_{i,j}^\Delta}^+ &= (v^2 - 1) \sum_{i < s < j} (v^{\tilde{f}_s} K_s K_j^{-1} u_{E_{k,s-j+l}^\Delta}^- u_{E_{i,s}^\Delta}^+ - v^{\tilde{g}_s} K_s K_i^{-1} u_{E_{s,j}^\Delta}^+ u_{E_{s+k-i,l}^\Delta}^-) \\ &\quad + v^{\tilde{f}_i} K_i K_j^{-1} u_{E_{k,i-j+l}^\Delta}^- - v^{\tilde{g}_j} K_i^{-1} K_j u_{E_{j+k-i,l}^\Delta}^-. \end{aligned}$$

(iii) If  $k - l > i - j$ , then

$$\begin{aligned} u_{E_{i,j}^\Delta}^+ u_{E_{k,l}^\Delta}^- - u_{E_{k,l}^\Delta}^- u_{E_{i,j}^\Delta}^+ &= v^{\tilde{f}_{k-l+j}} K_i K_j^{-1} u_{E_{i,k-l+j}^\Delta}^+ + (v^2 - 1) \sum_{k-l+j < s < j} v^{\tilde{f}_s} K_s K_j^{-1} u_{E_{k,s-j+l}^\Delta}^- u_{E_{i,s}^\Delta}^+ \\ &\quad - v^{\tilde{g}_{l-k+i}} K_i^{-1} K_j u_{E_{l-k+i,j}^\Delta}^+ - (v^2 - 1) \sum_{i < s < l-k+i} v^{\tilde{g}_s} K_i^{-1} K_s u_{E_{s,j}^\Delta}^+ u_{E_{s+k-i,l}^\Delta}^-. \end{aligned}$$

*Proof.* For convenience, we let  $u_{E_{s,s}^\Delta}^+ = u_{E_{s,s}^\Delta}^- = 1$  for  $s \in \mathbb{Z}$ . Let  $A = E_{i,j}^\Delta$  and  $B = E_{k,l}^\Delta$ . Recall from (5.0.2) the definition of  $L_{A,B}$  and  $R_{A,B}$ . Applying (3.4.2) gives that  $\tilde{K}^{\mathbf{d}(C)} = K^{\deg(u_C^+)} = K^{\text{ro}(C) - \text{co}(C)}$  for  $C \in \Theta_\Delta^+(n)$ . In particular we have  $\tilde{K}^{\mathbf{d}(E_{s,t}^\Delta)} = K_s K_t^{-1}$  for  $s < t$ . This together with 5.2 implies that

$$\begin{aligned} L_{A,A} &= v^{\tilde{f}_j} u_A^- u_A^+ + (v^2 - 1) \sum_{i < s < j} v^{\tilde{f}_s} K_s K_j^{-1} u_{E_{i,s}^\Delta}^- u_{E_{i,s}^\Delta}^+ + \frac{v^{\tilde{f}_i}}{v^2 - 1} K_i K_j^{-1}, \\ R_{A,A} &= v^{\tilde{g}_i} u_A^+ u_A^- + (v^2 - 1) \sum_{i < s < j} v^{\tilde{g}_s} K_s K_i^{-1} u_{E_{s,j}^\Delta}^+ u_{E_{s,j}^\Delta}^- + \frac{v^{\tilde{g}_j}}{v^2 - 1} K_i^{-1} K_j, \end{aligned}$$

and if  $A \neq B$ , then

$$L_{A,B} = \begin{cases} v^{f_j} u_B^- u_A^+ & \text{if } \bar{j} \neq \bar{l}, \\ v^{f_j} u_B^- u_A^+ + (v^2 - 1) \sum_{a < s < j} v^{f_s} K_s K_j^{-1} u_{E_{k,s-j+l}^\Delta}^- u_{E_{i,s}^\Delta}^+ + v^{f_a} K_a K_j^{-1} u_{E_{k,a-j+l}^\Delta}^- u_{E_{i,a}^\Delta}^+ & \text{if } \bar{j} = \bar{l}, \end{cases}$$

$$R_{A,B} = \begin{cases} v^{f_j} u_A^+ u_B^- & \text{if } \bar{i} \neq \bar{k}, \\ v^{f_j} u_A^+ u_B^- + (v^2 - 1) \sum_{i < s < b} v^{g_s} K_s K_i^{-1} u_{E_{s,j}^\Delta}^+ u_{E_{s+k-i,l}^\Delta}^- + v^{g_b} K_b K_i^{-1} u_{E_{b,j}^\Delta}^+ u_{E_{b+k-i,l}^\Delta}^- & \text{if } \bar{i} = \bar{k}, \end{cases}$$

where  $a = \max\{i, k - l + j\}$  and  $b = \min\{j, l - k + i\}$ . Combining this with 5.1 proves the assertion.  $\square$

## 6. THE CLASSICAL ( $v = 1$ ) CASE

Let  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  be the universal enveloping algebra of  $\widehat{\mathfrak{gl}}_n$ , where  $\widehat{\mathfrak{gl}}_n = \mathfrak{gl}_n(\mathbb{Q}) \otimes \mathbb{Q}[t, t^{-1}]$  is the loop algebra associated to the general linear Lie algebra  $\mathfrak{gl}_n(\mathbb{Q})$  over  $\mathbb{Q}$ . Using Hall algebras, we will construct the  $\mathbb{Z}$ -form  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  and prove in 6.5 that  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  is a  $\mathbb{Z}$ -subalgebra of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ . In addition, we will prove in 6.7 that the natural algebra homomorphism from  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  to  $\mathcal{S}_{\Delta}(n, r)_{\mathbb{Z}}$  is surjective, where  $\mathcal{S}_{\Delta}(n, r)_{\mathbb{Z}} = \mathcal{S}_{\Delta}(n, r) \otimes_{\mathbb{Z}} \mathbb{Z}$  is the affine Schur algebra over  $\mathbb{Z}$ .

Recall the set  $M_{\Delta n}(\mathbb{Q})$  defined in 1.1. We will identify  $\widehat{\mathfrak{gl}}_n$  with  $M_{\Delta n}(\mathbb{Q})$  via the following lie algebra isomorphism

$$M_{\Delta n}(\mathbb{Q}) \longrightarrow \widehat{\mathfrak{gl}}_n, \quad E_{i,j+ln}^\Delta \longmapsto E_{i,j} \otimes t^l, \quad 1 \leq i, j \leq n, l \in \mathbb{Z}.$$

Let  $\mathcal{U}^+(\widehat{\mathfrak{gl}}_n)$  (resp.,  $\mathcal{U}^-(\widehat{\mathfrak{gl}}_n)$ ,  $\mathcal{U}^0(\widehat{\mathfrak{gl}}_n)$ ) be the subalgebra of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  generated by  $E_{i,j}^\Delta$  (resp.,  $E_{j,i}^\Delta$ ,  $E_{i,i}^\Delta$ ), for  $1 \leq i \leq n$ ,  $j \in \mathbb{Z}$  and  $i < j$ . Then we have

$$(6.0.1) \quad \mathcal{U}(\widehat{\mathfrak{gl}}_n) = \mathcal{U}^+(\widehat{\mathfrak{gl}}_n) \otimes \mathcal{U}^0(\widehat{\mathfrak{gl}}_n) \otimes \mathcal{U}^-(\widehat{\mathfrak{gl}}_n),$$

Recall  $\mathfrak{D}_{\Delta}^+(n) = \text{span}_{\mathbb{Z}}\{\tilde{u}_A^+ \mid A \in \Theta_{\Delta}^+(n)\}$  and  $\mathfrak{D}_{\Delta}^-(n) = \text{span}_{\mathbb{Z}}\{\tilde{u}_A^- \mid A \in \Theta_{\Delta}^-(n)\}$ . Then  $\mathfrak{D}_{\Delta}^+(n)$  and  $\mathfrak{D}_{\Delta}^-(n)$  are all  $\mathcal{Z}$ -subalgebras of  $\mathfrak{D}_{\Delta}(n)$ . Note that  $\mathfrak{D}_{\Delta}^+(n) \cong \mathfrak{D}_{\Delta}^-(n)^{\text{op}} \cong \mathfrak{H}_{\Delta}(n)$ , where  $\mathfrak{H}_{\Delta}(n)$  is the Hall algebra over  $\mathcal{Z}$  associated with cyclic quivers  $\Delta(n)$ . Let  $\mathfrak{D}_{\Delta}^+(n)_{\mathbb{Q}} = \mathfrak{D}_{\Delta}^+(n) \otimes_{\mathbb{Z}} \mathbb{Q}$ ,  $\mathfrak{D}_{\Delta}^+(n)_{\mathbb{Z}} = \mathfrak{D}_{\Delta}^+(n) \otimes_{\mathbb{Z}} \mathbb{Z}$ ,  $\mathfrak{D}_{\Delta}^-(n)_{\mathbb{Q}} = \mathfrak{D}_{\Delta}^-(n) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\mathfrak{D}_{\Delta}^-(n)_{\mathbb{Z}} = \mathfrak{D}_{\Delta}^-(n) \otimes_{\mathbb{Z}} \mathbb{Z}$ , where  $\mathbb{Q}$  and  $\mathbb{Z}$  are regarded as  $\mathbb{Z}$ -modules by specializing  $v$  to 1. For  $A \in \Theta_{\Delta}^+(n)$  let  $u_{A,1}^+ = u_A^+ \otimes 1$  and  $u_{A,1}^- = u_A^- \otimes 1$ .

**Lemma 6.1** ([5, 6.1.2]). *There is a unique injective algebra homomorphism  $\theta^+ : \mathfrak{D}_{\Delta}^+(n)_{\mathbb{Q}} \rightarrow \mathcal{U}(\widehat{\mathfrak{gl}}_n)$  (resp.,  $\theta^- : \mathfrak{D}_{\Delta}^-(n)_{\mathbb{Q}} \rightarrow \mathcal{U}(\widehat{\mathfrak{gl}}_n)$ ) taking  $u_{E_{i,j}^\Delta,1}^+ \mapsto E_{i,j}^\Delta$  (resp.,  $u_{E_{i,j}^\Delta,1}^- \mapsto E_{j,i}^\Delta$ ) for all  $i < j$  such that  $\theta^+(\mathfrak{D}_{\Delta}^+(n)_{\mathbb{Q}}) = \mathcal{U}^+(\widehat{\mathfrak{gl}}_n)$  and  $\theta^-(\mathfrak{D}_{\Delta}^-(n)_{\mathbb{Q}}) = \mathcal{U}^-(\widehat{\mathfrak{gl}}_n)$ .*

We now use 6.1 to introduce the integral form  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  for  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ . Let  $\mathcal{U}_{\mathbb{Z}}^+(\widehat{\mathfrak{gl}}_n) = \theta^+(\mathfrak{D}_{\Delta}^+(n)_{\mathbb{Z}})$  and  $\mathcal{U}_{\mathbb{Z}}^-(\widehat{\mathfrak{gl}}_n) = \theta^-(\mathfrak{D}_{\Delta}^-(n)_{\mathbb{Z}})$ . Let  $\mathcal{U}_{\mathbb{Z}}^0(\widehat{\mathfrak{gl}}_n)$  be the  $\mathbb{Z}$ -submodule of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  spanned

by  $\prod_{1 \leq i \leq n} \binom{E_{i,i}^\Delta}{\lambda_i}$ , for  $\lambda \in \mathbb{N}_\Delta^n$ , where

$$\binom{E_{i,i}^\Delta}{\lambda_i} = \frac{E_{i,i}^\Delta(E_{i,i}^\Delta - 1) \cdots (E_{i,i}^\Delta - \lambda_i + 1)}{\lambda_i!}.$$

Let

$$\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n) = \mathcal{U}_\mathbb{Z}^+(\widehat{\mathfrak{gl}}_n) \mathcal{U}_\mathbb{Z}^0(\widehat{\mathfrak{gl}}_n) \mathcal{U}_\mathbb{Z}^-(\widehat{\mathfrak{gl}}_n) = \text{span}_\mathbb{Z} \left\{ w_A^+ \prod_{1 \leq i \leq n} \binom{E_{i,i}^\Delta}{\lambda_i} w_B^- \mid A, B \in \Theta_\Delta^+(n), \lambda \in \mathbb{N}_\Delta^n \right\},$$

where  $w_A^+ = \theta^+(u_{A,1}^+)$  and  $w_B^- = \theta^-(u_{B,1}^-)$ .

**Lemma 6.2.** *The set  $\{w_A^+ \prod_{1 \leq i \leq n} \binom{E_{i,i}^\Delta}{\lambda_i} w_B^- \mid A, B \in \Theta_\Delta^+(n), \lambda \in \mathbb{N}_\Delta^n\}$  forms a  $\mathbb{Z}$ -basis for  $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$  and  $\mathcal{U}(\widehat{\mathfrak{gl}}_n) \cong \mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n) \otimes_\mathbb{Z} \mathbb{Q}$ .*

*Proof.* By [15] and [13, 26.4] we conclude that the set  $\{\prod_{1 \leq i \leq n} \binom{E_{i,i}^\Delta}{\lambda_i} \mid \lambda \in \mathbb{N}_\Delta^n\}$  forms a  $\mathbb{Q}$ -basis for  $\mathcal{U}^0(\widehat{\mathfrak{gl}}_n)$ . Now the assertion follows from (6.0.1) and 6.1.  $\square$

To prove that  $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$  is a  $\mathbb{Z}$ -subalgebra of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ , we need some preparation. Recall  $\dot{\mathfrak{D}}_\Delta(n)$  defined in (4.0.1). Let  $\dot{\mathfrak{D}}_\Delta(n)_\mathbb{Q} = \dot{\mathfrak{D}}_\Delta(n) \otimes_\mathbb{Z} \mathbb{Q}$ . By 4.2,  $\dot{\mathfrak{D}}_\Delta(n)_\mathbb{Q}$  is a  $\mathbb{Q}$ -algebra. For  $A \in \Theta_\Delta^+(n)$  and  $\lambda \in \mathbb{Z}_\Delta^n$ , let  $u_{A,1}^+ = u_A^+ \otimes 1$ ,  $u_{A,1}^- = u_A^- \otimes 1$  and  $1_{\lambda,1} = 1_\lambda \otimes 1$ . By 5.3, we immediately get the following result.

**Lemma 6.3.** *Let  $i, j \in \mathbb{Z}$ ,  $i < j$  and  $k < l$ . The following formulas hold in  $\dot{\mathfrak{D}}_\Delta(n)_\mathbb{Q}$ .*

- (1) *If  $\bar{j} \neq \bar{l}$  and  $\bar{i} \neq \bar{k}$ , then  $1_{\lambda,1} u_{E_{i,j},1}^+ u_{E_{k,l},1}^- = 1_{\lambda,1} u_{E_{k,l},1}^- u_{E_{i,j},1}^+$ .*
- (2) *If  $\bar{j} = \bar{l}$  and  $\bar{i} \neq \bar{k}$ , then*

$$1_{\lambda,1} (u_{E_{i,j},1}^+ u_{E_{k,l},1}^- - u_{E_{k,l},1}^- u_{E_{i,j},1}^+) = \begin{cases} 1_{\lambda,1} u_{E_{k,i-j+l},1}^- & \text{if } k-l < i-j \\ 1_{\lambda,1} u_{E_{i,k-l+j},1}^+ & \text{if } k-l > i-j \end{cases}$$

- (3) *If  $\bar{j} \neq \bar{l}$  and  $\bar{i} = \bar{k}$ , then*

$$1_{\lambda,1} (u_{E_{i,j},1}^+ u_{E_{k,l},1}^- - u_{E_{k,l},1}^- u_{E_{i,j},1}^+) = \begin{cases} -1_{\lambda,1} u_{E_{j+k-i,l},1}^- & \text{if } k-l < i-j \\ -1_{\lambda,1} u_{E_{l-k+i,j},1}^+ & \text{if } k-l > i-j \end{cases}$$

- (4) *If  $\bar{j} = \bar{l}$  and  $\bar{i} = \bar{k}$ , then*

$$1_{\lambda,1} (u_{E_{i,j},1}^+ u_{E_{k,l},1}^- - u_{E_{k,l},1}^- u_{E_{i,j},1}^+) = \begin{cases} (\lambda_i - \lambda_j) 1_{\lambda,1} & \text{if } k-l = i-j \\ 1_{\lambda,1} (u_{E_{k,i-j+l},1}^- - u_{E_{j+k-i,l},1}^-) & \text{if } k-l < i-j \\ 1_{\lambda,1} (u_{E_{i,k-l+j},1}^+ - u_{E_{l-k+i,j},1}^+) & \text{if } k-l > i-j \end{cases}$$



Mimicking the construction of  $\widehat{\mathfrak{D}}_\Delta(n)$ , let  $\widehat{\mathfrak{D}}_\Delta(n)_\mathbb{Q}$  be the  $\mathbb{Q}$ -vector space of all formal (possibly infinite)  $\mathbb{Q}$ -linear combinations  $\sum_{A, B \in \Theta_\Delta^+(n), \lambda \in \mathbb{Z}_\Delta^n} \beta_{A, B, \lambda} u_{A, 1}^+ 1_{\lambda, 1} u_{B, 1}^-$  satisfying the property (F) with a similar multiplication. This is an associative  $\mathbb{Q}$ -algebra with an identity:  $\sum_{\lambda \in \mathbb{Z}_\Delta^n} 1_{\lambda, 1}$ . The algebra  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  is related to the algebra  $\widehat{\mathfrak{D}}_\Delta(n)_\mathbb{Q}$  in the following way (cf. 4.4).

**Proposition 6.4.** *There is an injective algebra homomorphism  $\varphi : \mathcal{U}(\widehat{\mathfrak{gl}}_n) \rightarrow \widehat{\mathfrak{D}}_\Delta(n)_\mathbb{Q}$  such that  $\varphi(E_{i, j}^\Delta) = \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{E_{i, j}, 1}^+ 1_{\lambda, 1}$ ,  $\varphi(E_{j, i}^\Delta) = \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{E_{j, i}, 1}^- 1_{\lambda, 1}$  and  $\varphi(E_{i, i}^\Delta) = \sum_{\lambda \in \mathbb{Z}_\Delta^n} \lambda_i 1_{\lambda, 1}$  for  $i < j$  and  $\lambda \in \mathbb{Z}_\Delta^n$ . Furthermore we have  $\varphi(w_A^+) = \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{A, 1}^+ 1_{\lambda, 1}$  and  $\varphi(w_A^-) = \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{A, 1}^- 1_{\lambda, 1}$  for  $A \in \Theta_\Delta^+(n)$ .*

*Proof.* For  $x, y \in \widehat{\mathfrak{D}}_\Delta(n)_\mathbb{Q}$  we set  $[x, y] = xy - yx$ . Then by (3.4.2) we have in  $\widehat{\mathfrak{D}}_\Delta(n)_\mathbb{Q}$ ,

$$(6.4.1) \quad \left[ \sum_{\lambda \in \mathbb{Z}_\Delta^n} \lambda_i 1_{\lambda, 1}, \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{E_{k, l}, 1}^+ 1_{\lambda, 1} \right] = \sum_{\lambda \in \mathbb{Z}_\Delta^n} \lambda_i u_{E_{k, l}, 1}^+ 1_{\lambda - e_k^\Delta + e_l^\Delta, 1} - \sum_{\lambda \in \mathbb{Z}_\Delta^n} \lambda_i u_{E_{k, l}, 1}^+ 1_{\lambda, 1} \\ = (\delta_{i, \bar{k}} - \delta_{i, \bar{l}}) \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{E_{k, l}, 1}^+ 1_{\lambda, 1}.$$

Applying [5, 6.1.1] yields

$$(6.4.2) \quad \left[ \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{E_{i, j}, 1}^+ 1_{\lambda, 1}, \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{E_{k, l}, 1}^+ 1_{\lambda, 1} \right] = \sum_{\lambda \in \mathbb{Z}_\Delta^n} (\delta_{\bar{j}, \bar{k}} u_{E_{i, l+j-k}, 1}^+ - \delta_{\bar{l}, \bar{i}} u_{E_{k, j+l-i}, 1}^+) 1_{\lambda, 1}, \\ \left[ \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{E_{i, j}, 1}^- 1_{\lambda, 1}, \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{E_{k, l}, 1}^- 1_{\lambda, 1} \right] = \sum_{\lambda \in \mathbb{Z}_\Delta^n} (\delta_{\bar{i}, \bar{l}} u_{E_{k+i-l, j}, 1}^- - \delta_{\bar{k}, \bar{j}} u_{E_{i+k-j, l}, 1}^-) 1_{\lambda, 1}.$$

for  $i < j$  and  $k < l$ . Furthermore, it is easy to see that  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  has a presentation with generators  $E_{i, j}^\Delta$  ( $1 \leq i \leq n$ ,  $j \in \mathbb{Z}$ ), and relations:

- (a)  $[E_{i, i}^\Delta, E_{k, l}^\Delta] = (\delta_{i, \bar{k}} - \delta_{i, \bar{l}}) E_{k, l}^\Delta$ .
- (b)  $[E_{i, j}^\Delta, E_{k, l}^\Delta] = \delta_{\bar{j}, \bar{k}} E_{i, l+j-k}^\Delta - \delta_{\bar{l}, \bar{i}} E_{k, j+l-i}^\Delta$  for  $i \neq j$  and  $k \neq l$ .

This, together with 6.3, (6.4.1) and (6.4.2), implies that there is an algebra homomorphism  $\varphi : \mathcal{U}(\widehat{\mathfrak{gl}}_n) \rightarrow \widehat{\mathfrak{D}}_\Delta(n)_\mathbb{Q}$  defined by sending  $E_{i, j}^\Delta$  to  $\sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{E_{i, j}, 1}^+ 1_{\lambda, 1}$ ,  $E_{j, i}^\Delta$  to  $\sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{E_{j, i}, 1}^- 1_{\lambda, 1}$ , and  $E_{i, i}^\Delta$  to  $\sum_{\lambda \in \mathbb{Z}_\Delta^n} \lambda_i 1_{\lambda, 1}$ .

Using an argument similar to the proof of 4.4, we can show that there is an injective  $\mathbb{Q}$ -algebra homomorphism  $\rho^+ : \mathfrak{D}_\Delta^+(n)_\mathbb{Q} \rightarrow \widehat{\mathfrak{D}}_\Delta(n)_\mathbb{Q}$  (resp.,  $\rho^- : \mathfrak{D}_\Delta^-(n)_\mathbb{Q} \rightarrow \widehat{\mathfrak{D}}_\Delta(n)_\mathbb{Q}$ ) taking  $u_{A, 1}^+ \mapsto \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{A, 1}^+ 1_{\lambda, 1}$  (resp.,  $u_{A, 1}^- \mapsto \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{A, 1}^- 1_{\lambda, 1}$ ) for  $A \in \Theta_\Delta^+(n)$ . Since  $\varphi \circ \theta^\pm(u_{E_{i, j}, 1}^\pm) = \rho^\pm(u_{E_{i, j}, 1}^\pm)$  and  $\mathfrak{D}_\Delta^\pm(n)_\mathbb{Q}$  is generated by  $u_{E_{i, j}, 1}^\pm$  for  $i < j$ , we see that  $\varphi \circ \theta^\pm = \rho^\pm$  and hence

$$\varphi(w_A^\pm) = \varphi \circ \theta^\pm(u_{A, 1}^\pm) = \rho^\pm(u_{A, 1}^\pm) = \sum_{\lambda \in \mathbb{Z}_\Delta^n} u_{A, 1}^\pm 1_{\lambda, 1}$$

for  $A \in \Theta_\Delta^+(n)$ .

Finally, we prove that  $\varphi$  is injective. Assume

$$x = \sum_{A,B \in \Theta_{\Delta}^+(n), \mathbf{j} \in \mathbb{N}_{\Delta}^n} k_{A,B,\mathbf{j}} w_A^+ \prod_{1 \leq i \leq n} (E_{i,i}^{\Delta})^{j_i} w_B^- \in \ker(\varphi),$$

where  $k_{A,B,\mathbf{j}} \in \mathbb{Q}$ . Then

$$\varphi(x) = \sum_{A,B \in \Theta_{\Delta}^+(n), \mu \in \mathbb{Z}_{\Delta}^n} \left( \sum_{\mathbf{j} \in \mathbb{N}_{\Delta}^n} k_{A,B,\mathbf{j}} \mu^{\mathbf{j}} \right) u_{A,1}^+ 1_{\mu,1} u_{B,1}^- = 0,$$

where  $\mu^{\mathbf{j}} = \mu_1^{j_1} \cdots \mu_n^{j_n}$ . This implies that  $\sum_{\mathbf{j} \in \mathbb{N}_{\Delta}^n} k_{A,B,\mathbf{j}} \mu^{\mathbf{j}} = 0$  for all  $A, B, \mathbf{j}$ . By [5, 6.3.3], we have  $\det(\mu^{\mathbf{j}})_{\mu, \mathbf{j} \in \mathbb{Z}_l^n} \neq 0$  for any  $l \geq 1$ , where  $\mathbb{Z}_l^n = \{\lambda \in \mathbb{Z}^n \mid 0 \leq \lambda_i \leq l-1, \forall i\}$ . It follows that  $k_{A,B,\mathbf{j}} = 0$  for all  $A, B, \mathbf{j}$  and hence  $x = 0$ . The proof is completed.  $\square$

We will identify  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$  with the subalgebra  $\varphi(\mathcal{U}(\widehat{\mathfrak{gl}}_n))$  of  $\widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Q}}$  via  $\varphi$ , and hence identify  $w_A^{\pm}$  with  $\sum_{\lambda \in \mathbb{Z}_{\Delta}^n} u_{A,1}^{\pm} 1_{\lambda,1}$  for  $A \in \Theta_{\Delta}^+(n)$ , etc.

Let  $\dot{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Z}} = \dot{\mathfrak{D}}_{\Delta}(n) \otimes_{\mathbb{Z}} \mathbb{Z}$  and let  $\widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Z}}$  be the  $\mathbb{Z}$ -submodule of  $\widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Q}}$  consisting of those elements  $\sum_{A,B \in \Theta_{\Delta}^+(n), \lambda \in \mathbb{Z}_{\Delta}^n} \beta_{A,B,\lambda} u_{A,1}^+ 1_{\lambda,1} u_{B,1}^-$  in  $\widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Q}}$  with  $\beta_{A,B,\lambda} \in \mathbb{Z}$  for all  $A, B, \lambda$ . Then by 4.2,  $\widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Z}}$  is a  $\mathbb{Z}$ -subalgebra of  $\widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Q}}$ . We can now prove that  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  is a  $\mathbb{Z}$ -subalgebra of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ .

**Theorem 6.5.** *We have  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n) = \widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Z}} \cap \mathcal{U}(\widehat{\mathfrak{gl}}_n)$ . In particular,  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$  is a  $\mathbb{Z}$ -subalgebra of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ .*

*Proof.* Clearly  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n) \subseteq \widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Z}} \cap \mathcal{U}(\widehat{\mathfrak{gl}}_n)$ . On the other hand, if  $x \in \widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Z}} \cap \mathcal{U}(\widehat{\mathfrak{gl}}_n)$ , then by 6.2, we may write

$$x = \sum_{\substack{A,B \in \Theta_{\Delta}^+(n) \\ \lambda \in \mathbb{N}_{\Delta}^n}} k_{A,B,\lambda} w_A^+ \prod_{1 \leq i \leq n} \begin{pmatrix} E_{i,i}^{\Delta} \\ \lambda_i \end{pmatrix} w_B^- = \sum_{\substack{A,B \in \Theta_{\Delta}^+(n) \\ \mu \in \mathbb{Z}_{\Delta}^n}} \left( \sum_{\lambda \in \mathbb{N}_{\Delta}^n} k_{A,B,\lambda} \begin{pmatrix} \mu \\ \lambda \end{pmatrix} \right) u_{A,1}^+ 1_{\mu,1} u_{B,1}^-,$$

where  $k_{A,B,\lambda} \in \mathbb{Q}$  and  $\begin{pmatrix} \mu \\ \lambda \end{pmatrix} = \prod_{1 \leq i \leq n} \begin{pmatrix} \mu_i \\ \lambda_i \end{pmatrix}$ . Since  $x \in \widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Z}}$  we have

$$k_{A,B,\mu} + \sum_{\substack{\lambda \in \mathbb{N}_{\Delta}^n, \sigma(\lambda) < \sigma(\mu) \\ \lambda_i \leq \mu_i, \forall i}} k_{A,B,\lambda} \begin{pmatrix} \mu \\ \lambda \end{pmatrix} = \sum_{\lambda \in \mathbb{N}_{\Delta}^n} k_{A,B,\lambda} \begin{pmatrix} \mu \\ \lambda \end{pmatrix} \in \mathbb{Z}$$

for  $A, B \in \Theta_{\Delta}^+(n)$  and  $\mu \in \mathbb{N}_{\Delta}^n$ , where, as before,  $\sigma(\mu) = \sum_{1 \leq i \leq n} \mu_i$ . Using induction on  $\sigma(\mu)$ , we conclude that  $k_{A,B,\mu} \in \mathbb{Z}$  for all  $A, B \in \Theta_{\Delta}^+(n)$  and  $\mu \in \mathbb{N}_{\Delta}^n$ , and hence  $x \in \mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n)$ . This proves the first assertion. Since  $\widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Z}}$  is a  $\mathbb{Z}$ -subalgebra of  $\widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Q}}$  by 4.2, we conclude that  $\mathcal{U}_{\mathbb{Z}}(\widehat{\mathfrak{gl}}_n) = \widehat{\mathfrak{D}}_{\Delta}(n)_{\mathbb{Z}} \cap \mathcal{U}(\widehat{\mathfrak{gl}}_n)$  is a  $\mathbb{Z}$ -subalgebra of  $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ .  $\square$

Finally we will establish affine Schur–Weyl reciprocity at the integral level. Let  $\mathcal{S}_{\Delta}(n, r)_{\mathbb{Q}} = \mathcal{S}_{\Delta}(n, r) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\mathcal{S}_{\Delta}(n, r)_{\mathbb{Z}} = \mathcal{S}_{\Delta}(n, r) \otimes_{\mathbb{Z}} \mathbb{Z}$ . For  $A \in \Theta_{\Delta}^{\pm}(n)$  and  $\lambda \in \Lambda_{\Delta}(n, r)$ , let  $A(\mathbf{0}, r)_1 =$

$A(\mathbf{0}, r) \otimes 1 \in \mathcal{S}_\Delta(n, r)_\mathbb{Q}$  and  $[\text{diag}(\lambda)]_1 = [\text{diag}(\lambda)] \otimes 1 \in \mathcal{S}_\Delta(n, r)_\mathbb{Q}$ . The following result is due to [5, 6.1.3 and 6.1.4] (cf. [23]).

**Lemma 6.6.** *There is a surjective algebra homomorphism  $\eta_r : \mathcal{U}(\widehat{\mathfrak{gl}}_n) \rightarrow \mathcal{S}_\Delta(n, r)_\mathbb{Q}$  such that  $\eta_r(E_{i,j}^\Delta) = E_{i,j}^\Delta(\mathbf{0}, r)_1$  for  $i \neq j$  and  $\eta_r(E_{i,i}^\Delta) = \sum_{\lambda \in \Lambda_\Delta(n, r)} \lambda_i [\text{diag}(\lambda)]_1$ . Furthermore we have  $\eta_r(w_A^+) = A(\mathbf{0}, r)_1$  and  $\eta_r(w_A^-) = {}^t A(\mathbf{0}, r)_1$  for  $A \in \Theta_\Delta^+(n)$ .*

**Theorem 6.7.** *The restriction of  $\eta_r$  to  $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$  gives a surjective  $\mathbb{Z}$ -algebra homomorphism  $\eta_r : \mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n) \rightarrow \mathcal{S}_\Delta(n, r)_\mathbb{Z}$ .*

*Proof.* Since  $\eta_r(E_{i,i}^\Delta) = \sum_{\mu \in \Lambda_\Delta(n, r)} \mu_i [\text{diag}(\mu)]_1$  for  $1 \leq i \leq n$ , by (3.0.3), we conclude that

$$\eta_r \left( \prod_{1 \leq i \leq n} \binom{E_{i,i}^\Delta}{\lambda_i} \right) = \sum_{\substack{\mu \in \Lambda_\Delta(n, r) \\ \mu_i \geq \lambda_i, 1 \leq i \leq n}} \binom{\mu}{\lambda} [\text{diag}(\mu)]_1$$

for  $\lambda \in \mathbb{N}_\Delta^n$ . It follows that  $\eta_r \left( \prod_{1 \leq i \leq n} \binom{E_{i,i}^\Delta}{\lambda_i} \right) = [\text{diag}(\lambda)]_1$  for  $\lambda \in \Lambda_\Delta(n, r)$ . This, together with 6.6 implies that

$$\eta_r(\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)) = \text{span}_\mathbb{Z} \{ A^+(\mathbf{0}, r)_1 [\text{diag}(\lambda)]_1 A^-(\mathbf{0}, r)_1 \mid A \in \Theta_\Delta^\pm(n), \lambda \in \Lambda_\Delta(n, r) \}.$$

Combining this with 3.2 and 6.5 proves the assertion.  $\square$

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